Approximation Algorithms for Continuous Clustering and Facility Location Problems

Deeparnab Chakrabarty, Maryam Negahbani, **Ankita Sarkar**

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Continuous center-based k-clustering

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Input.

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- Clients $C \subseteq X$, $|C| = n$
- $k \in \mathbb{N}$

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  - k-median: $\sum_{v \in C} d(v, S)$
  - k-means: $\sum_{v \in C} d(v, S)^2$

Continuous vs. discrete

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- Discrete best:
  - k-median: 2.675
    [Byrka, Pensyl, Rybicki, Srinivasan, and Trinh, TALG 2017]
  - k-means: 9
    [Kanungo, Mount, Netanyahu, Piatko, Silverman, and Wu, ComGeo 2004] ...

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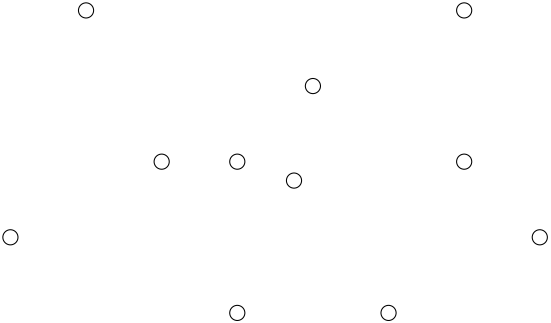

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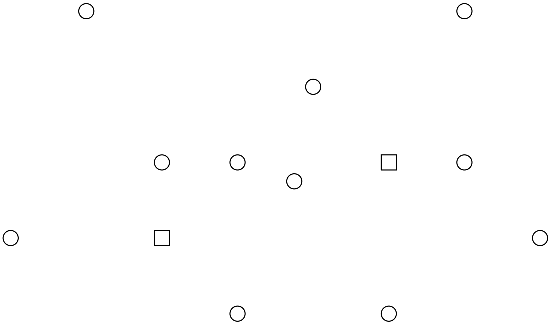

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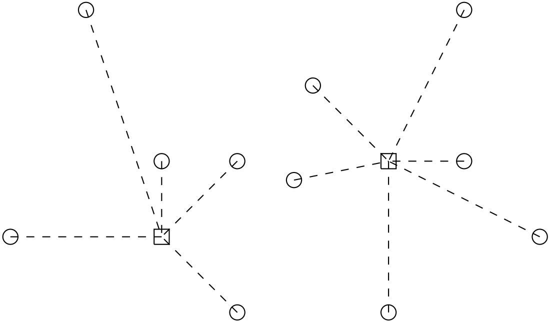

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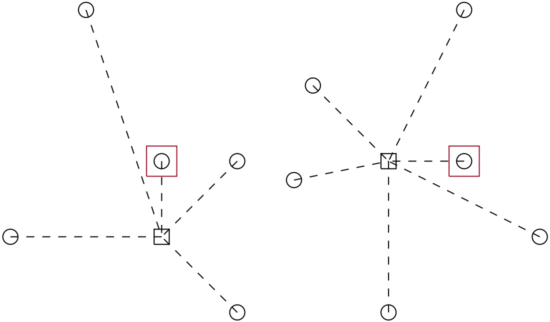

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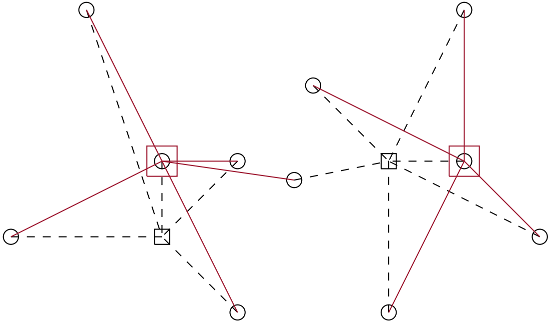

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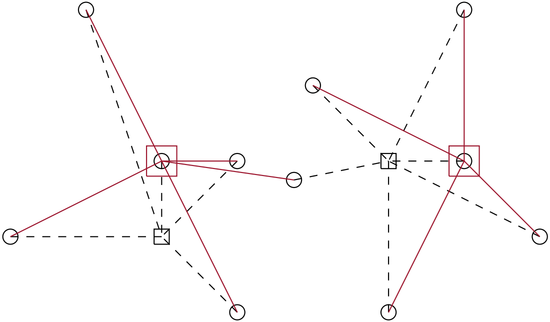

k-median: $2\times$

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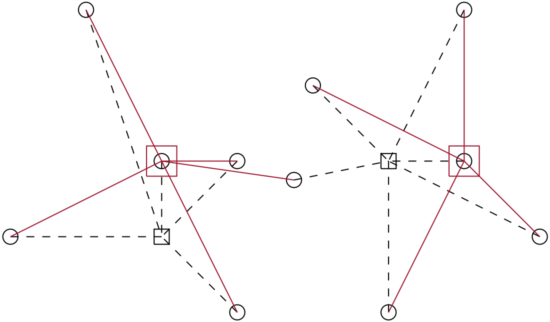

k-median: $2\times$
k-means: $4\times$

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- α-approx for discrete $\implies$ $\beta \cdot \alpha$-approx for continuous.
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- General (X, d)?

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- Fair-k-median, k-center with outliers : **Yes**.

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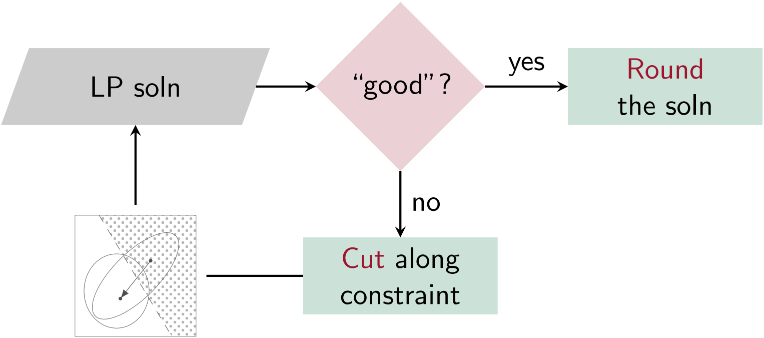

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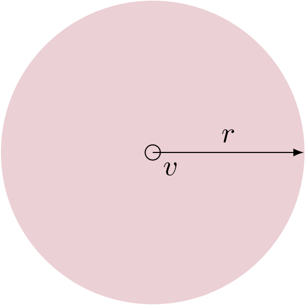

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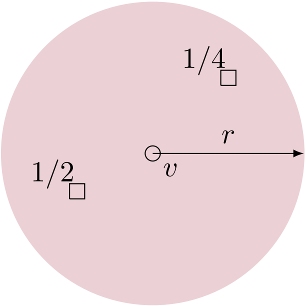

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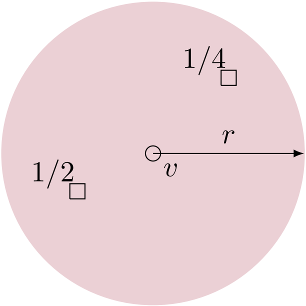

$y(v, r) = 3/4$

New LP: Consistency constraints

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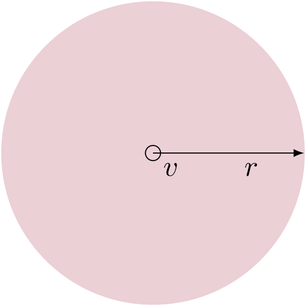

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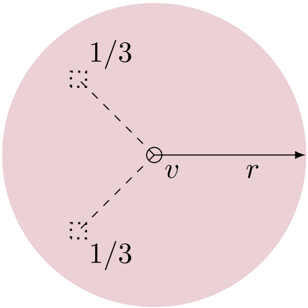

$$\underbrace{y(v,r)}_{2/3}$$

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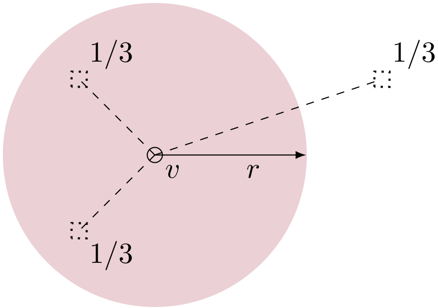

$$\underbrace{1 - y(v, r)}_{1/3}$$

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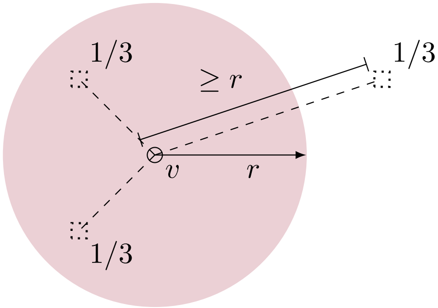

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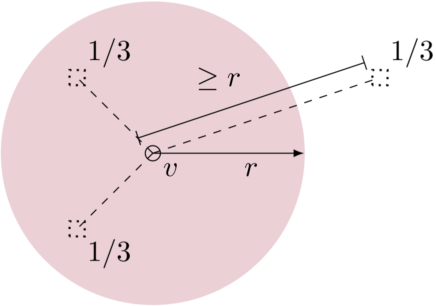

$$C_v \geq r \underbrace{(1 - y(v, r))}_{1/3}$$

New LP: Disjoint balls constraints

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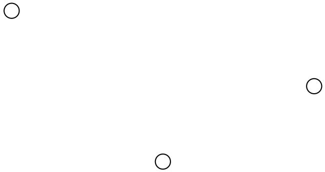
$$k = 1$$



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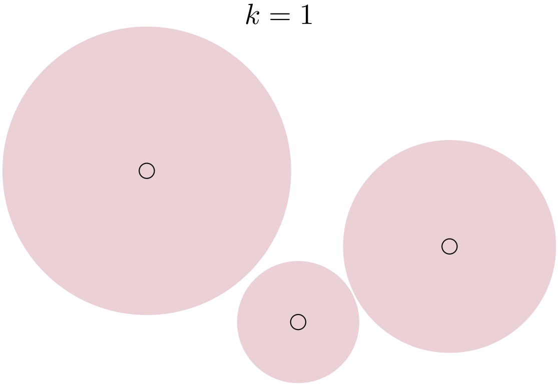

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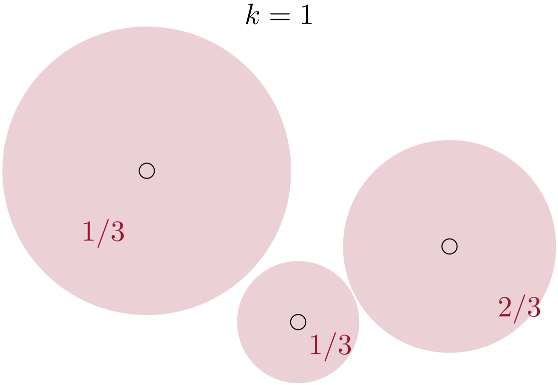

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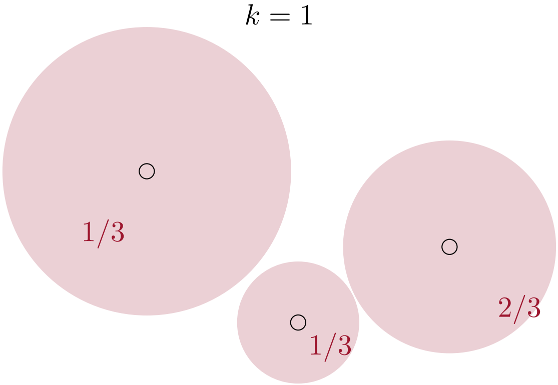

For $\mathcal{B} \subseteq \{B(v,r)\}_{v \in C, r \in \mathbb{R}}$, pairwise disjoint

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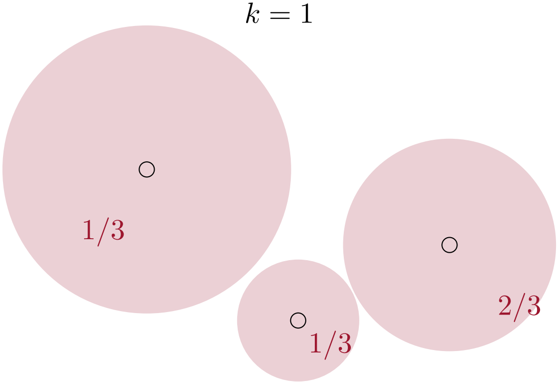
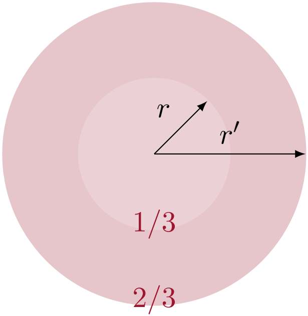


For $\mathcal{B} \subseteq \{B(v,r)\}_{v \in C, r \in \mathbb{R}}$, pairwise disjoint

$$\sum_{B \in \mathcal{B}} y(B) \leq k$$

New LP: Monotonicity constraints

$$y(v, r) \leq y(v, r'), \forall v \in C, r \leq r' \in \mathbb{R}$$



New LP: Constraints

Consistency. $\forall v \in C, r \in \mathbb{R}$,
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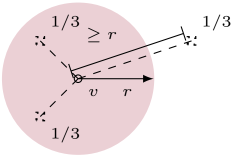
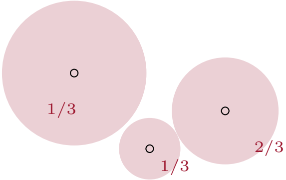


Disjoint balls. $\forall$ pairwise disj. $\mathcal{B}$,
$\sum_{B \in \mathcal{B}} y(B) \leq k$



Monotonicity. $\forall v \in C, r \leq r' \in \mathbb{R}$,
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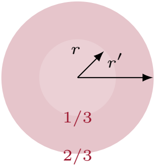

Satisfying disjoint balls constraints via round-or-cut

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(C, y)

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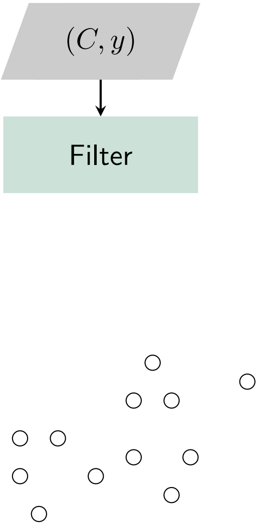

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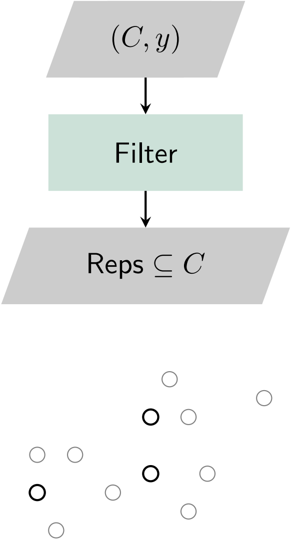

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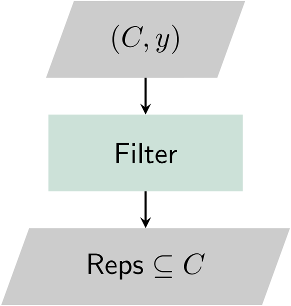

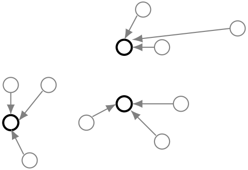

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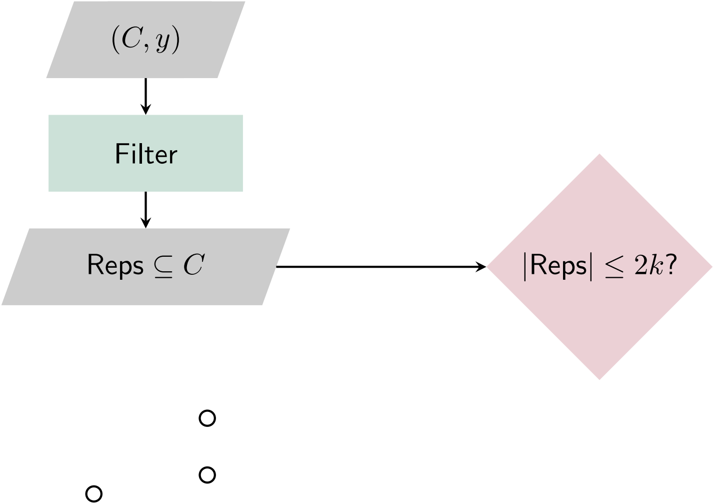

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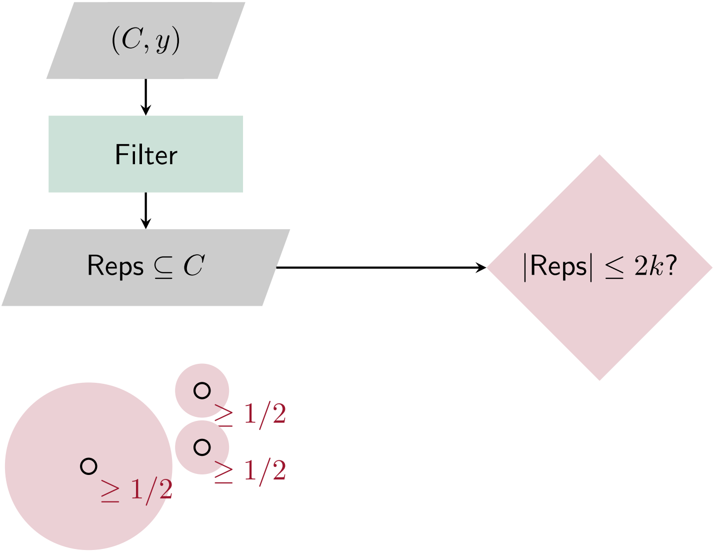

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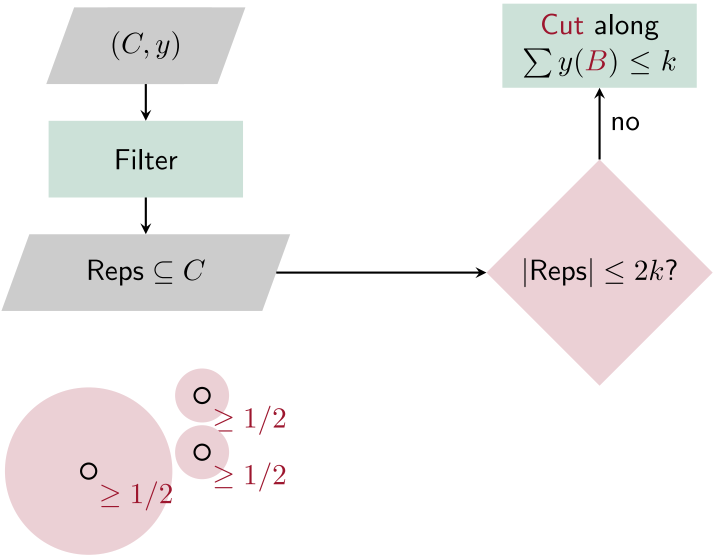

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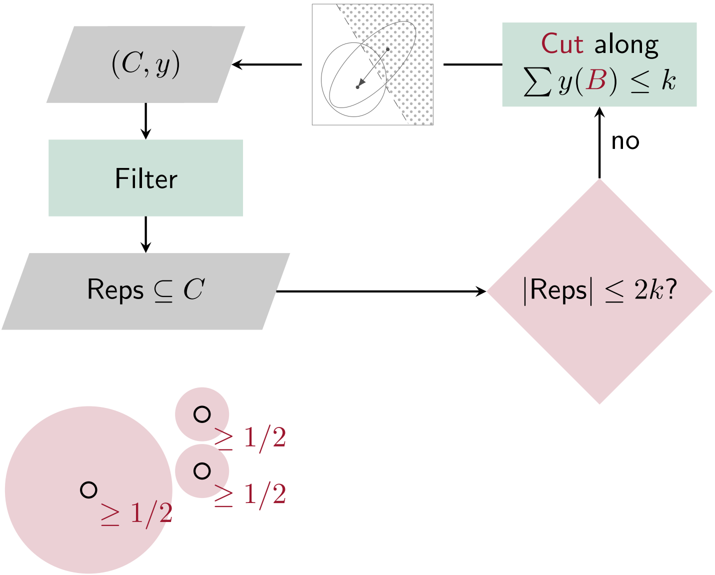

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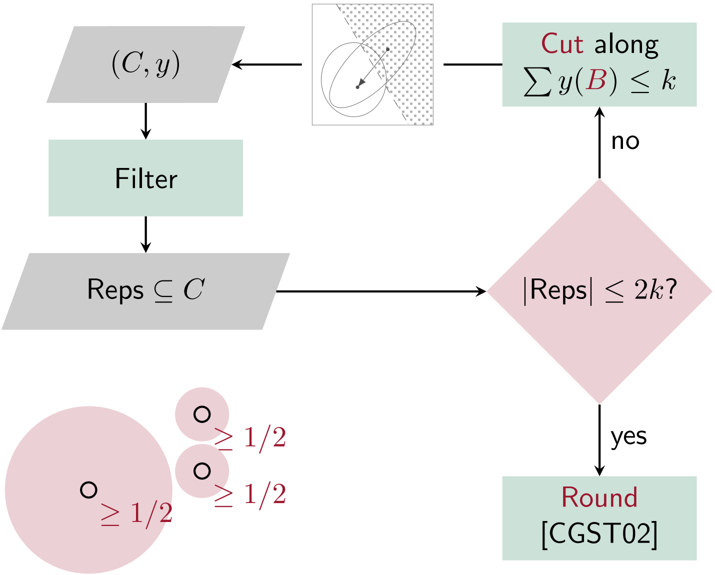

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Question

Better than 5.35-approx for continuous k-median?

Summary

- Continuous vs discrete: β gap?
- New LP: C_v, $y(v, r)$
- Round-or-cut
- Open: continuous k-median