Popular Matchings under Lower Quotas and their relationship to Stability under Classifications

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Contents

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Assumption : Strict preference lists, i.e. a vertex v has preference list of the form $(E(v), <_v)$.

Defining “Happiness”

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- no pair has incentive to deviate from the assigned matching.

Blocking Pairs and Stability

Definition

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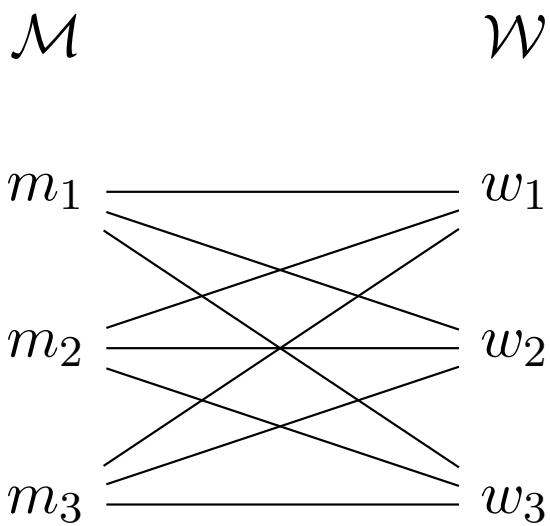
Definition

A matching M is said to be stable if there are no blocking pairs with respect to M.

Example



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m_1 ——— w_1

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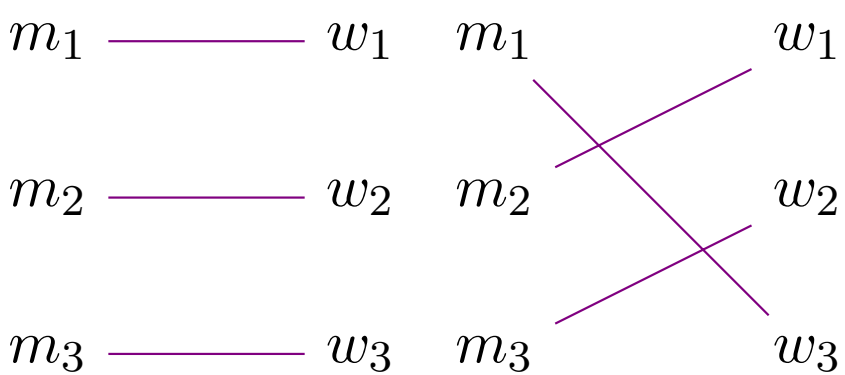

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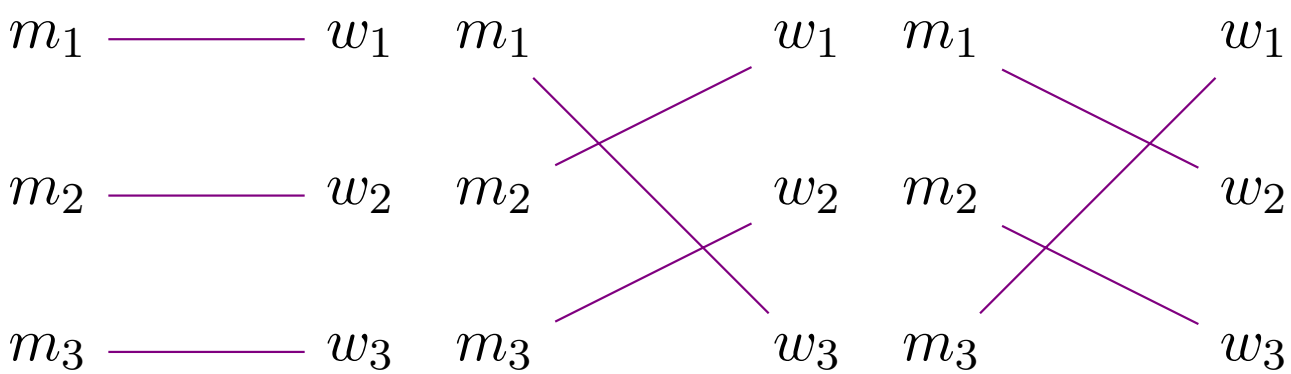

Finding Stable Matchings

[Gale and Shapley, 1962] : showed existence of stable matchings via an algorithm to find them.

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- Each man proposes to women, in the order most to least preferred, until he finds a match or exhausts his preference list.
- Each woman provisionally accepts the first proposal she receives. Thereafter, she compares incoming proposals with her existing partner, and rejects the less preferred man.

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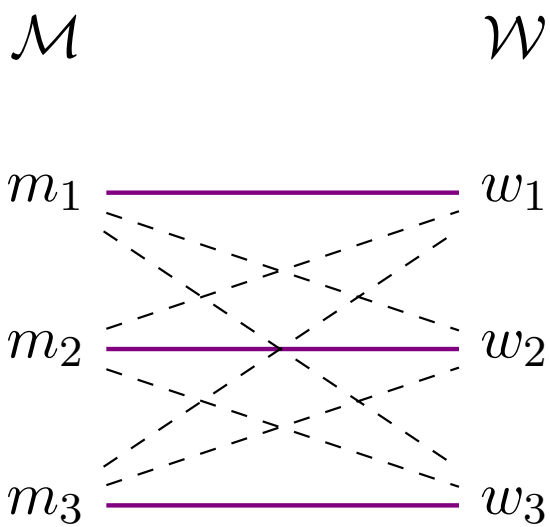

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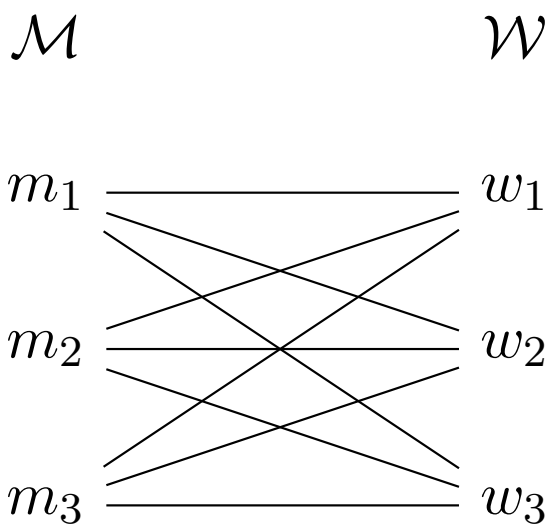

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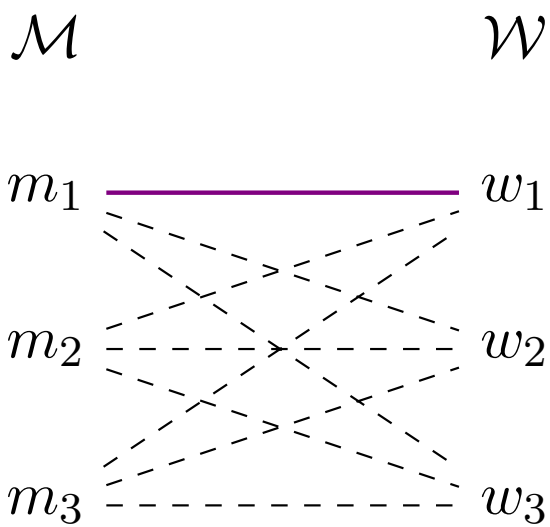

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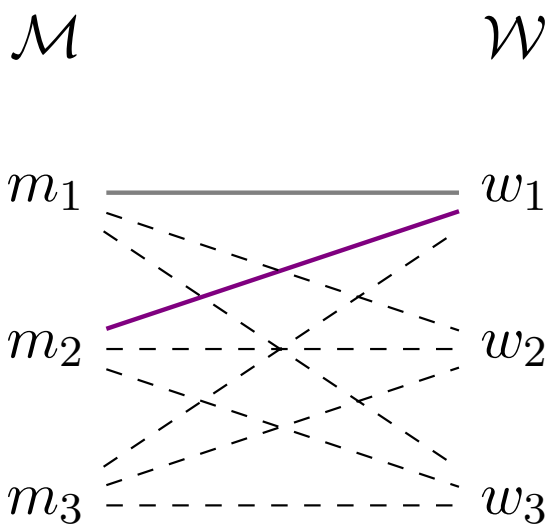

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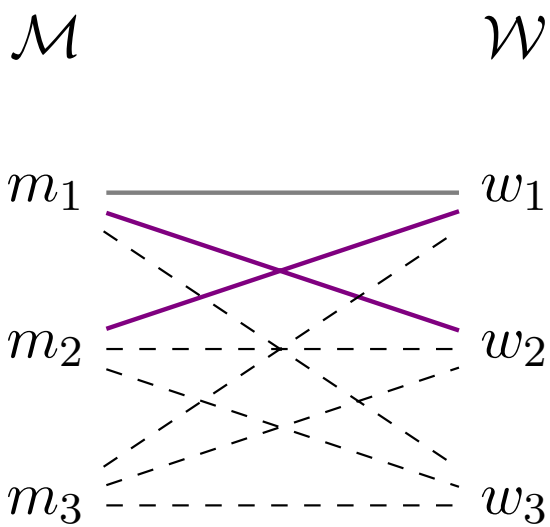

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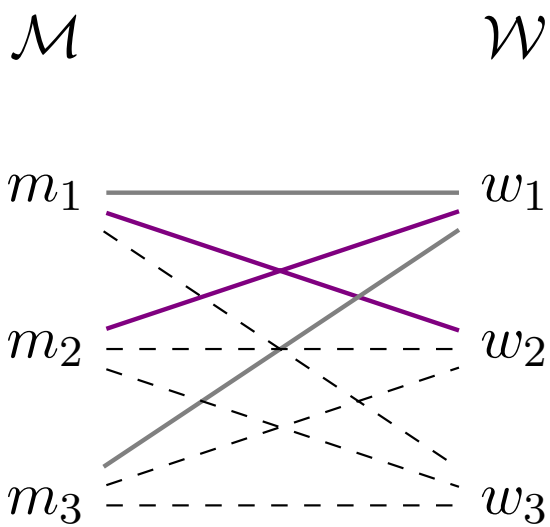

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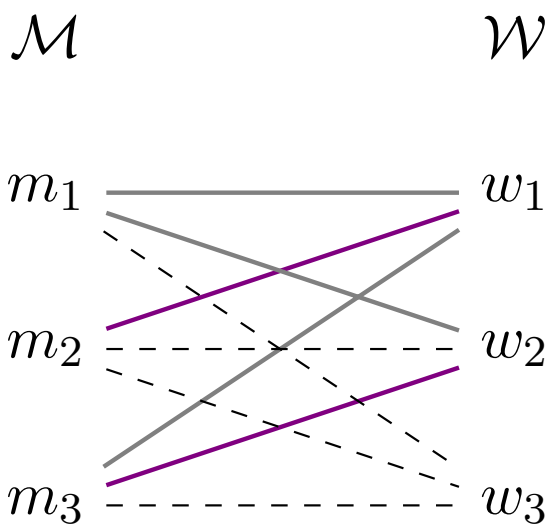

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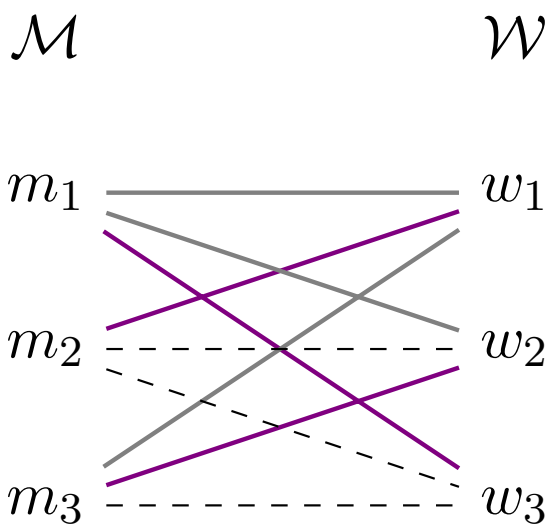

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Definition

A matching is $M \subseteq E$ such that $\forall h \in \mathcal{H}, |M(h)| \leq q^+(h)$ and $\forall r \in \mathcal{R}, |M(r)| = 1$.

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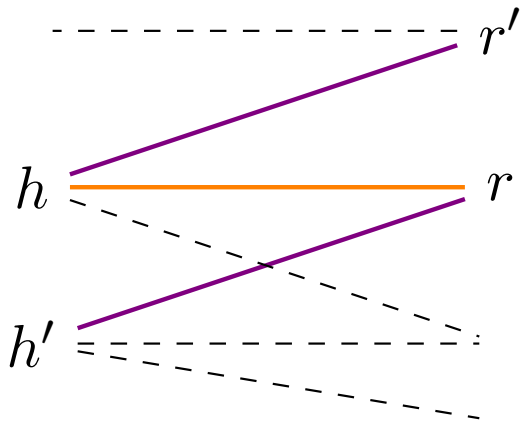

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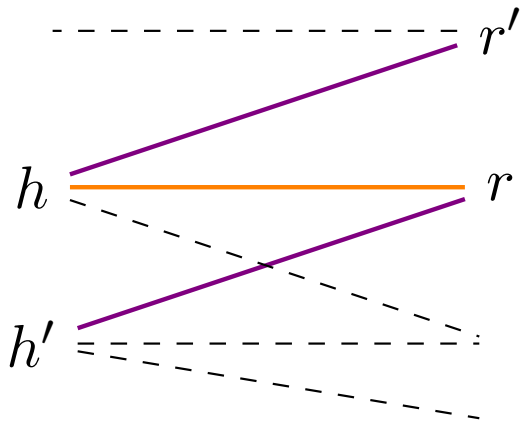

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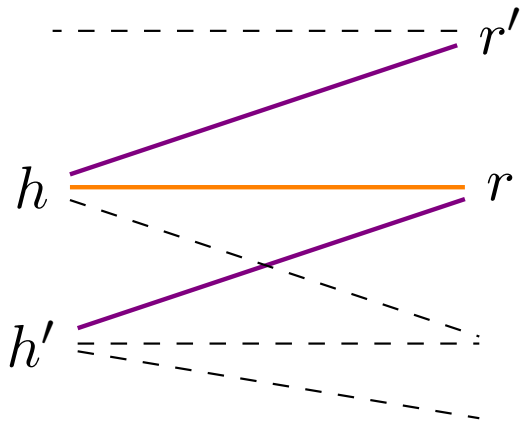

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- But partners of residents only improve. So $h' \geq_r h'' >_r h$, which is a contradiction.

Complexity

Each hospital proposes at most once to each of its neighbour residents. Hence, the Gale-Shapley algorithm runs in $\mathcal{O}(|E|)$ time.

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- *Undersubscribed hospitals are matched to the same set of residents in every stable matching.*

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Theorem (Rural Hospitals Theorem, [Roth, 1986])

- *All stable matchings have the same size.*
- *Undersubscribed hospitals are matched to the same set of residents in every stable matching.*

This means that if the Gale-Shapley output is small, all stable matchings are small. They can in fact be as small as $1/2$ the size of a maximum matching.

Example : Small stable matching

$m_1 : w_1 \ w_2$ $\qquad$ $w_1 : m_1 \ m_2$

$m_2 : w_1$ $\qquad$ $w_2 : m_1$

The Gale-Shapley output is $\{(m_1, w_1)\}$ while the maximum matching is of size 2.

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- **Stability :** no pair has incentive to deviate from the assigned matching.

Defining "Happiness"

- **Stability :** no pair has incentive to deviate from the assigned matching.
- **Popularity :** in a comparison with any other matching, a majority of vertices prefer this matching.

Voting between Matchings

Definition

For a vertex v, its vote is defined as follows :
$vote_v : E(v) \times E(v) \to \{-1, 1\}$, where

$$vote_v(u, w) = \begin{cases} 1 & u >_v w \\ -1 & u <_v w \\ 0 & u = w \end{cases}$$

Voting between Matchings

We want to have an election between matchings where each vertex votes. But how do hospitals compare their multiple neighbours?

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Definition

For a vertex v, wlog, $|M(v) \setminus M'(v)| \leq |M'(v) \setminus M(v)|$. Fix an arbitrary bijection between $M(v) \setminus M'(v)$ and an arbitrary subset of the same size in $M'(v) \setminus M(v)$. This bijection is $\mathbf{corr}_v|_{M(v) \setminus M'(v)}$ and for any remaining vertex u in $M'(v) \setminus M(v)$, $\mathbf{corr}_v(u) = \perp$.

Voting between Matchings

Once we fix an arbitrary correspondence as above for each hospital, we can calculate each vertex's vote between two matchings :

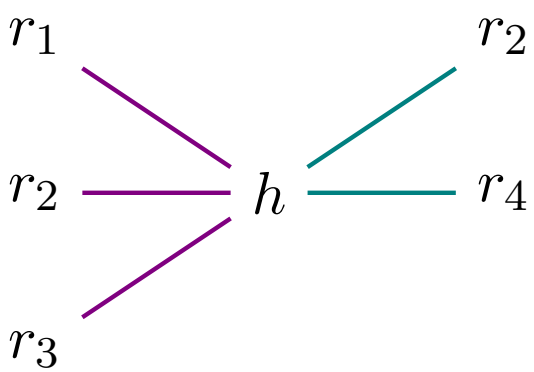
Definition

$$vote_v(M, M') = \sum_{u \in M(v) \setminus M'(v)} vote_v(u, corr_v(u))$$

Here we assume that $vote_v(u, \perp)$ is always 1 for any u.

Example

$M_1(h)$ $M_2(h)$



$h : r_1\ r_2\ r_4\ r_3$

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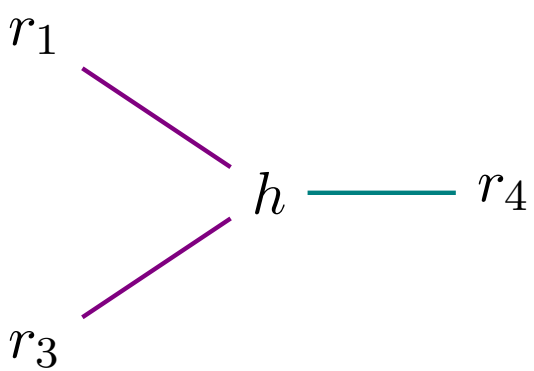

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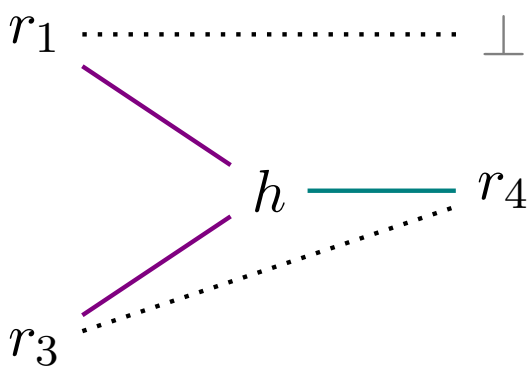

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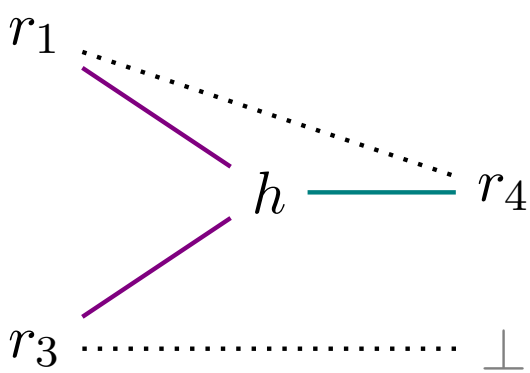

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$$vote_h(M_1, M_2) = 1 + 1 = 2$$

Popularity

Now we can formalise popularity :

Definition

M' is more popular than M (denoted $M' \succ M$) if

$$\sum_{v \in \mathcal{H} \uplus \mathcal{R}} vote_v(M', M) > 0$$

M is popular among a set of matchings $\mathcal{M} \subseteq 2^E$ if no matching in $\mathcal{M}$ is more popular than M.

Finding Popular Matchings

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Finding Popular Matchings

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- **Task :** Find maximum cardinality popular matchings.

Modifying Gale-Shapley

Algorithm in [Huang and Kavitha, 2011].

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Idea :

- Usual round of proposals as per Gale-Shapley.
- Undersubscribed hospitals propose again with higher priority, where residents always prefer higher priority hospitals.

Example

In the earlier example:

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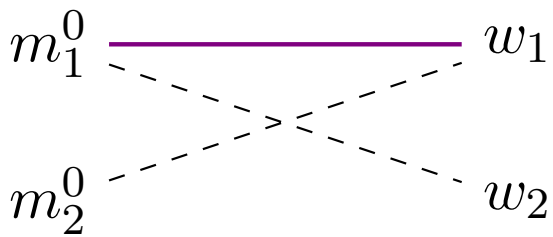

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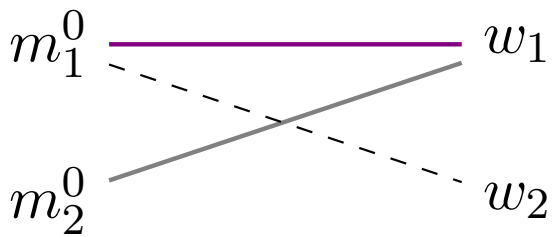

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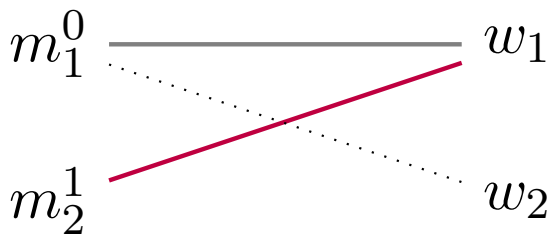

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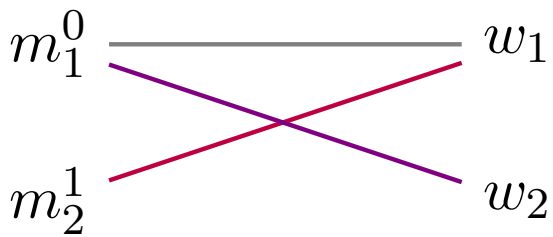

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- Votes gained : men matched at priority 1, women matched with priority-0 men.

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- Our matching be M, other matching be N. Consider alternating paths and cycles (according to $\mathbf{corr}$) in $M \oplus N$.
- Label N-edges in the above paths and cycles as per the vote of each vertex v in $vote_v(N, M)$.
- Opposing votes along an edge cancel; we show that there are at most as many $(+1, +1)$ edges as $(-1, -1)$ edges.

Correctness

Proof sketch :

- M^0 : Men matched at priority 0; M^1 : Men matched at priority 1
- W_0 : Unmatched women and women matched to priority-0 men; W_1 : women matched to priority-1 men.

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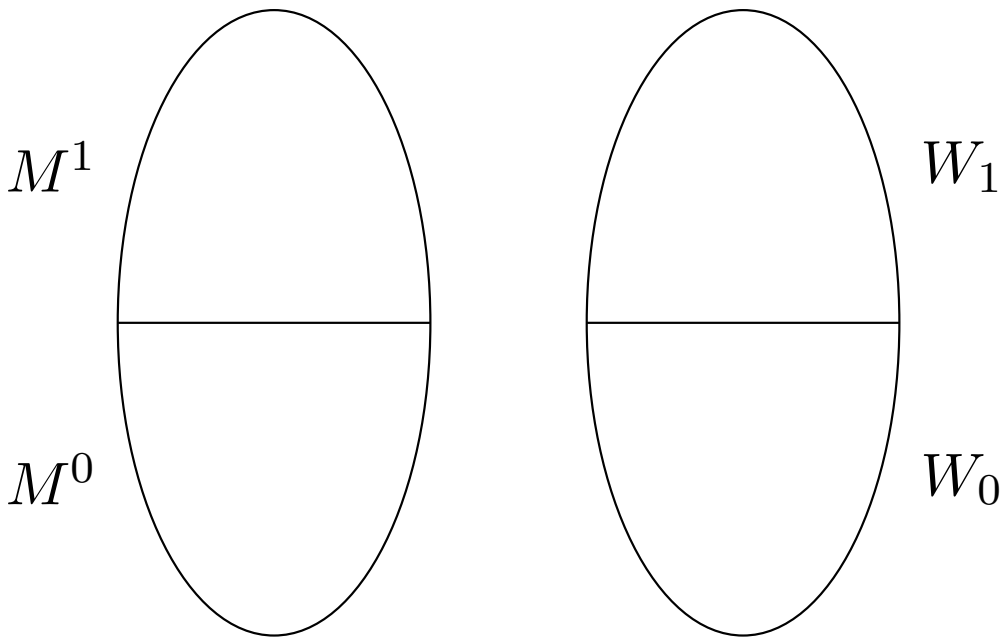

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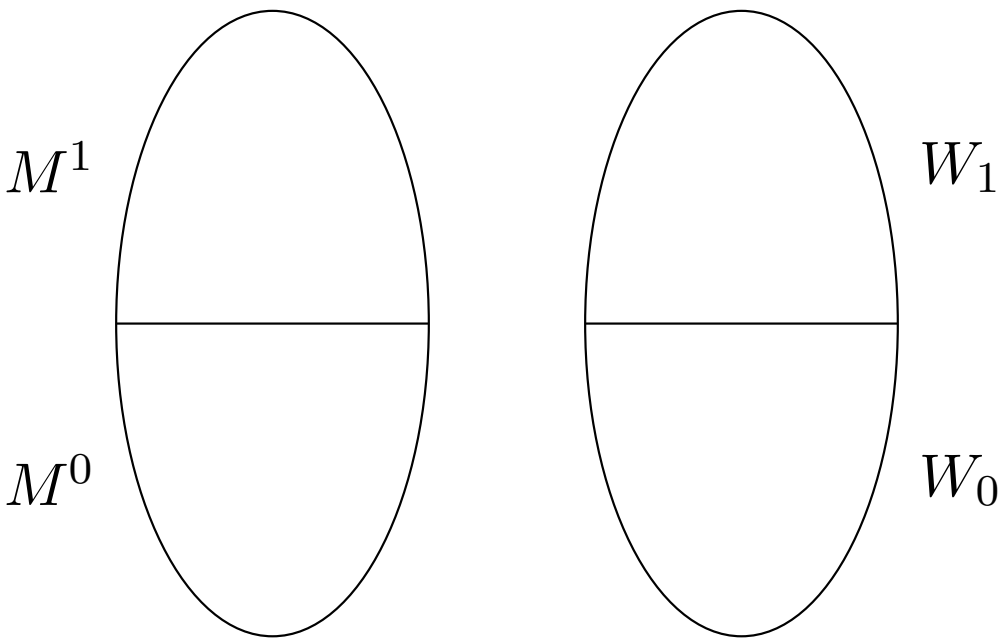

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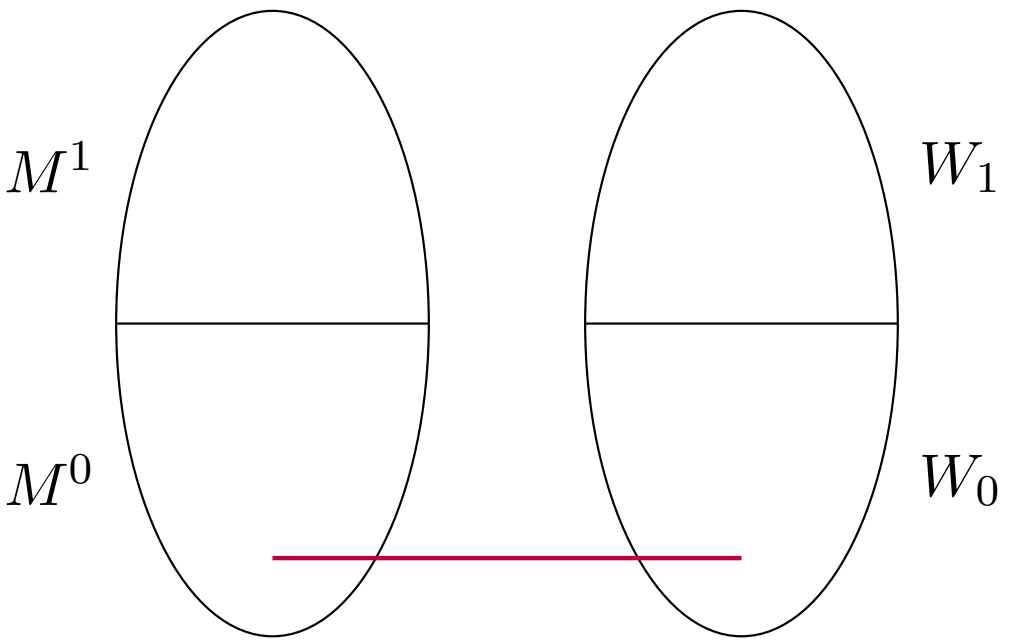

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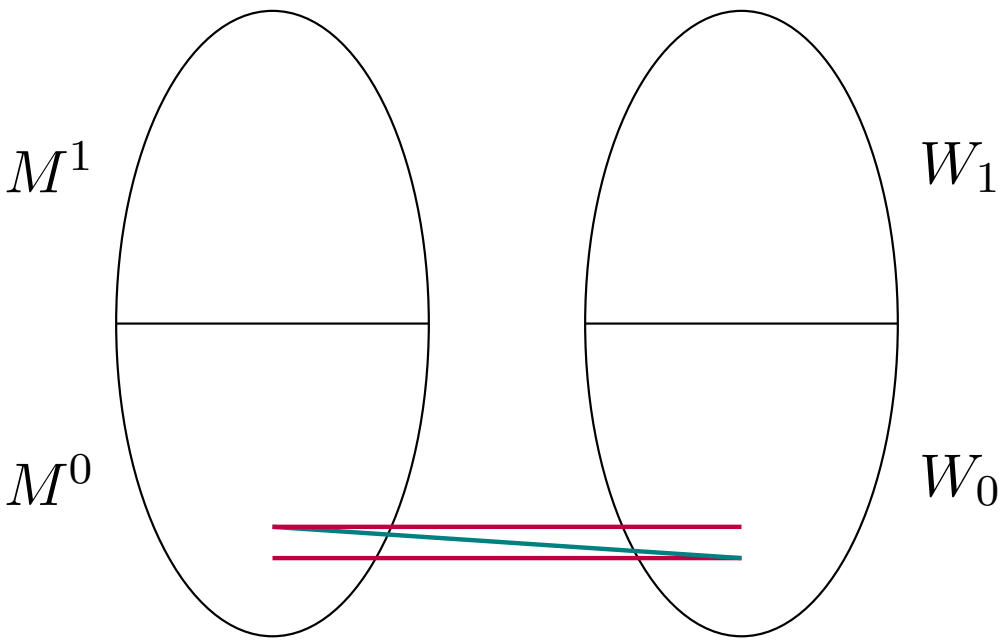

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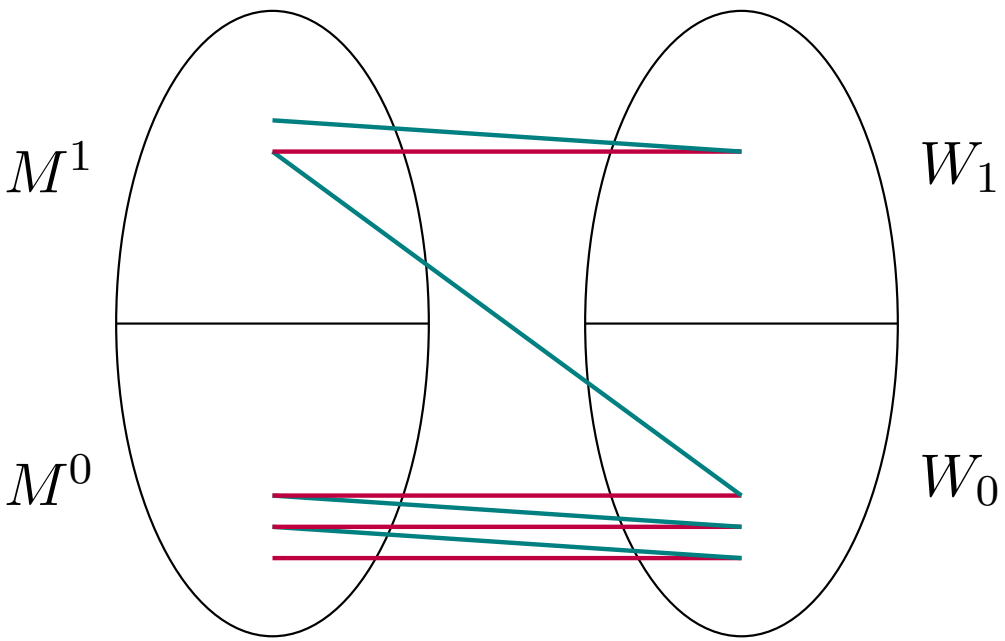

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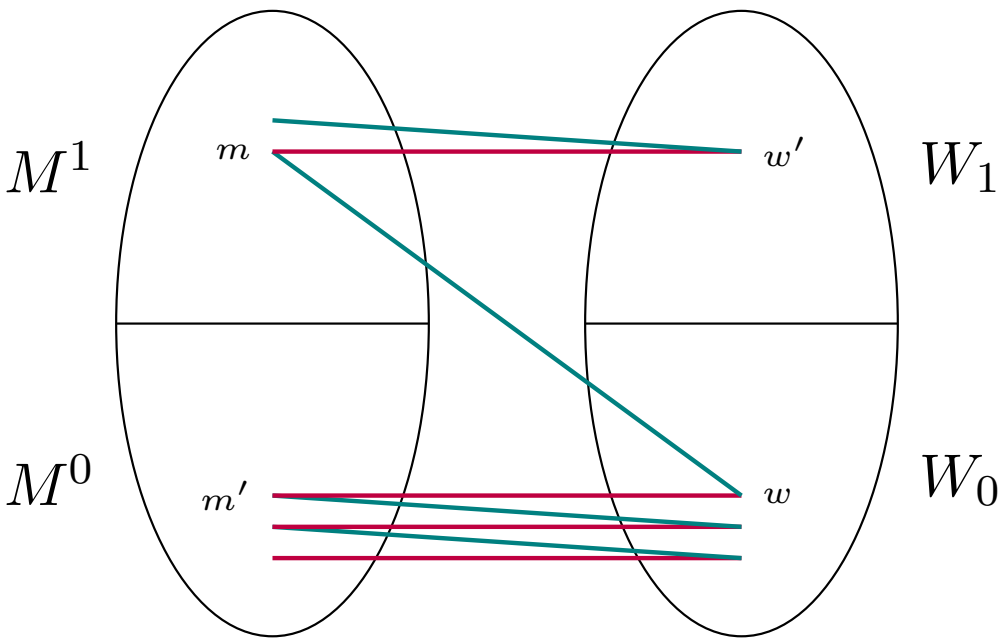

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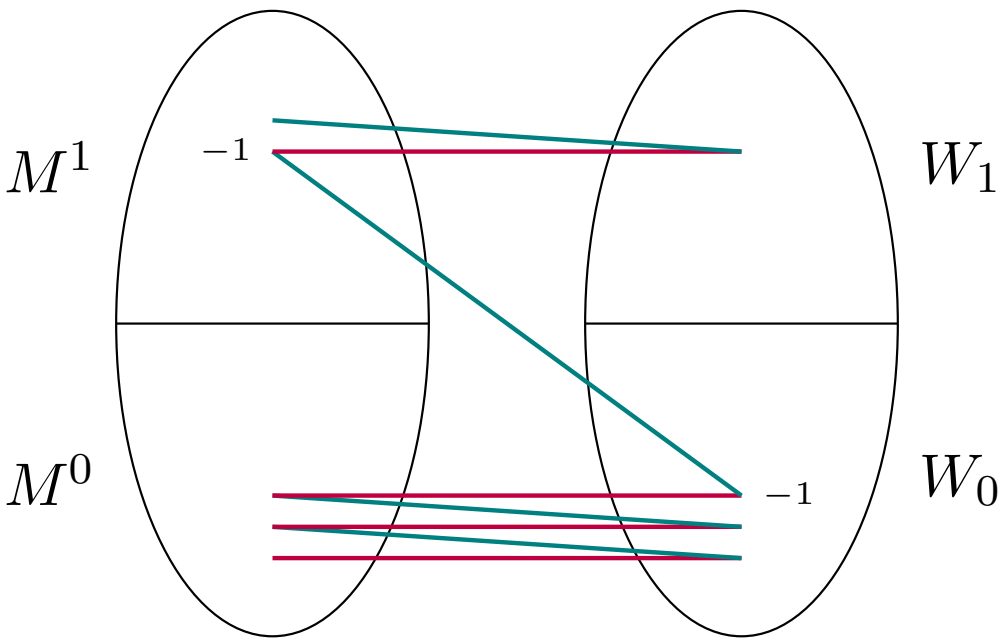

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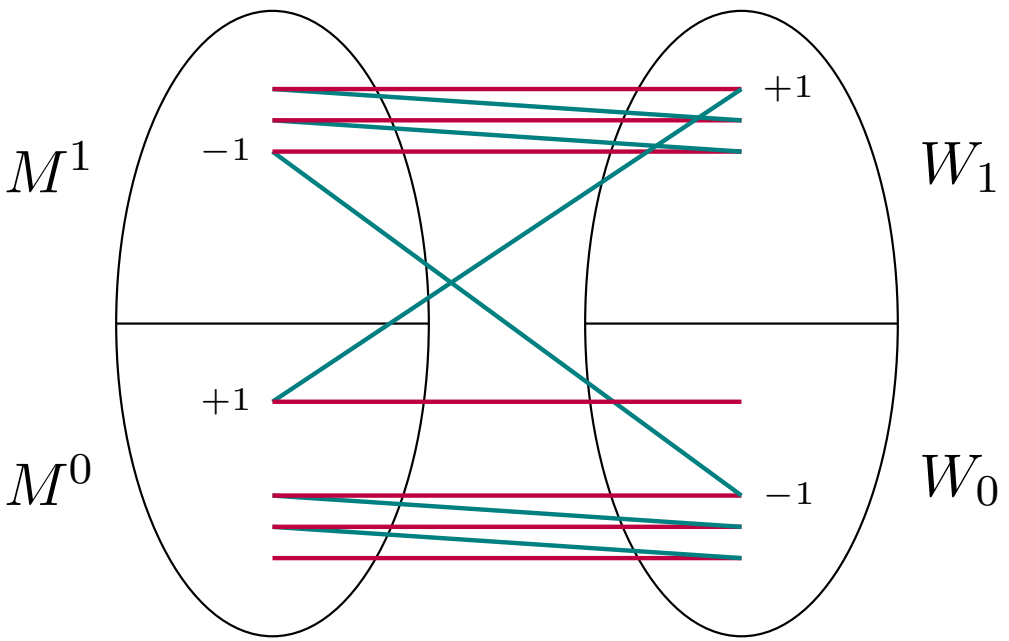

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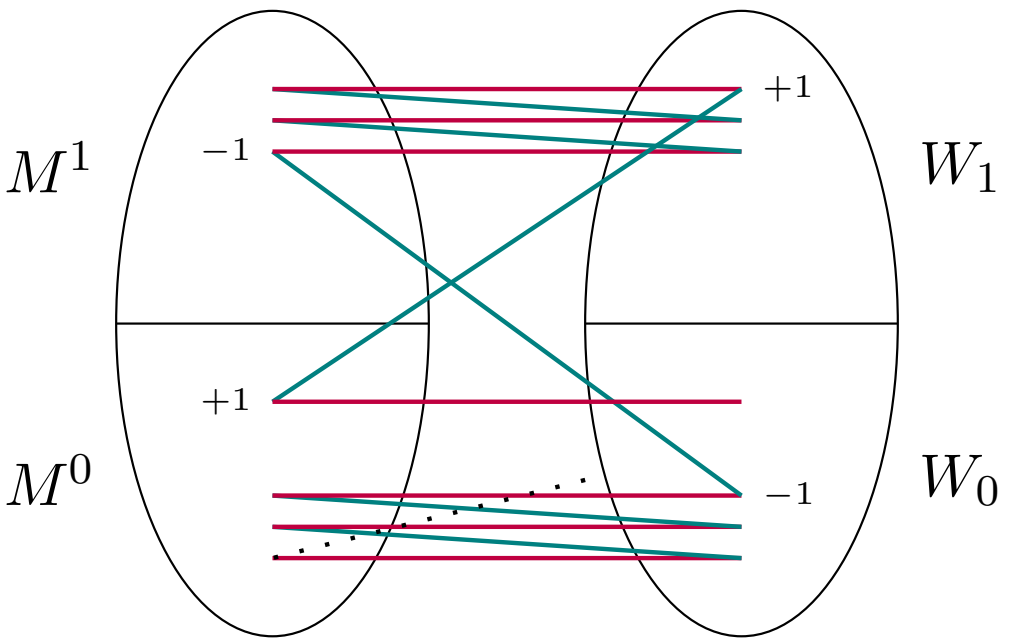

Reduction to Stability

Construction : we make two copies of each hospital h, and add dummy residents $d_h^1, \ldots, d_h^{q^+(h)}$.

Original instance	New instance
$h : r_1 \ \ldots \ r_k$	$h^0 : r_1 \ \ldots, r_k \ d_h^1 \ \ldots \ d_h^{q^+(h)}$ $h^1 : d_h^1 \ \ldots \ d_h^{q^+(h)} \ r_1 \ \ldots, r_k$
$r : h_1 \ \ldots \ h_k$	$r : h_1^1 \ \ldots \ h_k^1 \ h_1^0 \ \ldots \ h_k^0$ $d_h^i : h^0 \ h^1$

Example

Transformed instance :

$$m_1^0 : w_1\ w_2\ d_{m_1}^1 \qquad w_1 : m_1^1\ m_2^1\ m_1^0\ m_2^0$$
$$m_1^1 : d_{m_1}^1\ w_1\ w_2 \qquad w_2 : m_1^1\ m_1^0$$
$$m_2^0 : w_1\ d_{m_2}^1 \qquad d_{m_1}^1 : m_1^0\ m_1^1$$
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Transformed instance :

$$
\begin{array}{ll}
m_1^0 : w_1\ w_2\ d_{m_1}^1 & w_1 : m_1^1\ m_2^1\ m_1^0\ m_2^0 \\
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m_2^1 : d_{m_2}^1\ w_1 & d_{m_2}^1 : m_2^0\ m_2^1
\end{array}
$$

Gale-Shapley output is $\{(m_1^0, w_2), (m_1^1, d_{m_1}^1), (m_2^0, d_{m_2}^1), (m_2^1, w_1)\}$.
This gives the final matching $\{(m_1, w_2), (m_2, w_1)\}$.

Contents

Hospital-Residents under Lower Quotas

Input :

- Usual hospital-residents instance : $G = (\mathcal{H} \uplus \mathcal{R}, E)$; $\forall v, (E(v), <_v)$; $q^+ : \mathcal{H} \to \mathbb{N}$.
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Question : Can we find a feasible stable matching?

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Task : Find a matching that respects lower quotas (is feasible), and also maximises "happiness".
Question : Can we find a feasible stable matching?
Answer : Not always.

Example

$$h_1 : r_1 \ r_2 \ r_3 \qquad r_1 : h_1$$
$$h_2 : r_2 \ r_3 \qquad r_2 : h_1 \ h_2$$
$$h_3 : r_3 \qquad r_3 : h_1 \ h_2 \ h_3$$

Hospitals h_1, h_2, h_3 have upper quotas $3, 2, 1$ respectively. All lower quotas are 1.

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Hospitals h_1, h_2, h_3 have upper quotas $3, 2, 1$ respectively. All lower quotas are 1.
Gale-Shapley matches all three residents with h_1. This is infeasible. By Rural Hospitals Theorem, then all stable matchings are infeasible.

Popularity among Feasible Matchings

Algorithm by [Nasre and Nimbhorkar, 2017] :

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Question : Is M feasible?
Answer : If there exists a feasible matching, then M is feasible.

Correctness

Proof sketch :
If there is a deficient hospital, we track an alternating path starting from it, counting hospitals in that path.

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If there is a deficient hospital, we track an alternating path starting from it, counting hospitals in that path.
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Thus our choice of μ will give a contradiction.

Complexity

Observation : Feasible instance $\implies \mu - 2 \leq |\mathcal{R}|$. We can assume that trivially infeasible instances that violate this are ruled out before using the algorithm.

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The new edge set is of size $\mu\,|E| = \mathcal{O}(|\mathcal{R}|\,|E|)$.

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$$h_1 : r_1\ r_2\ r_3 \qquad\qquad r_1 : h_1$$
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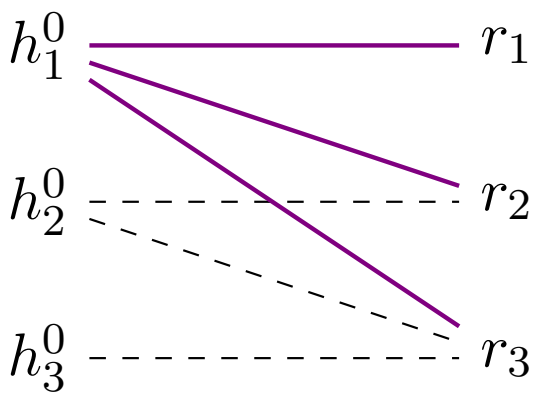

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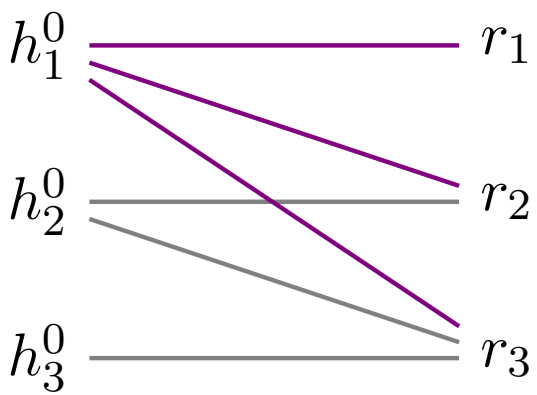

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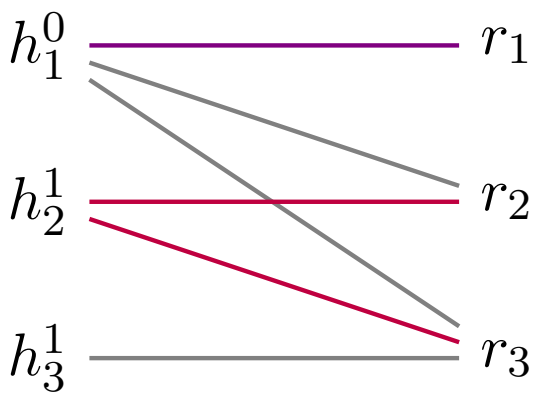

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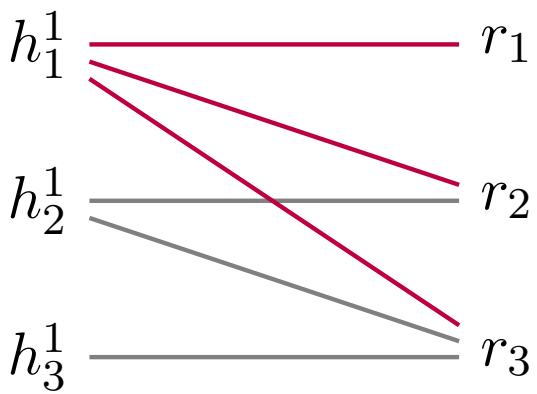

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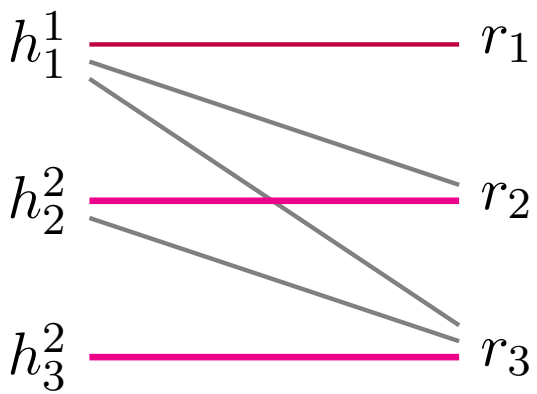

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Strategy :

- Allow repeated proposals as before.
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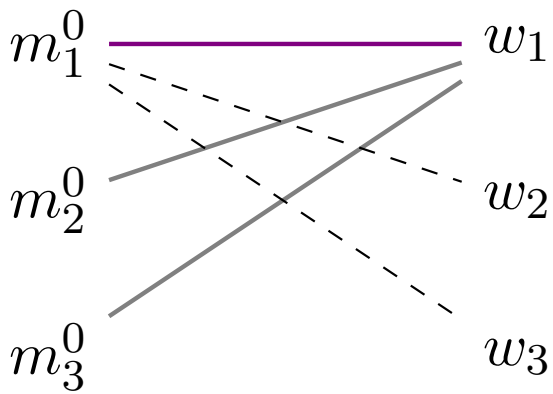

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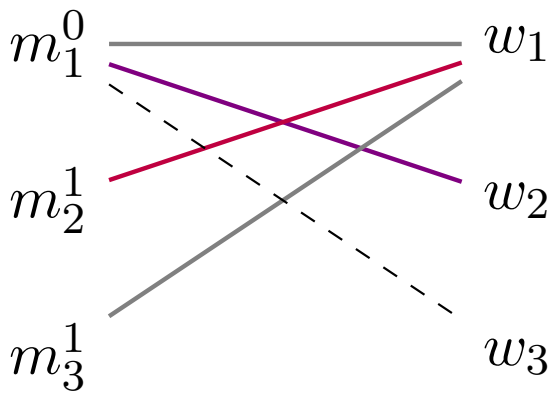

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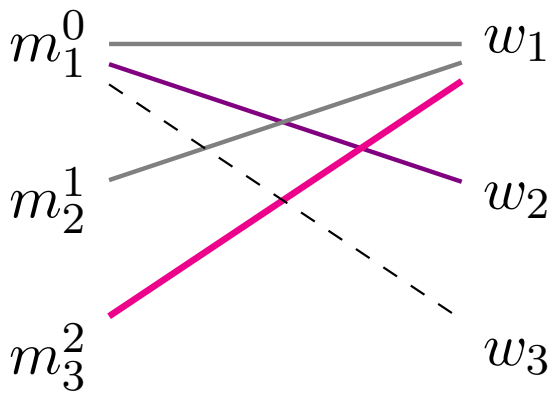

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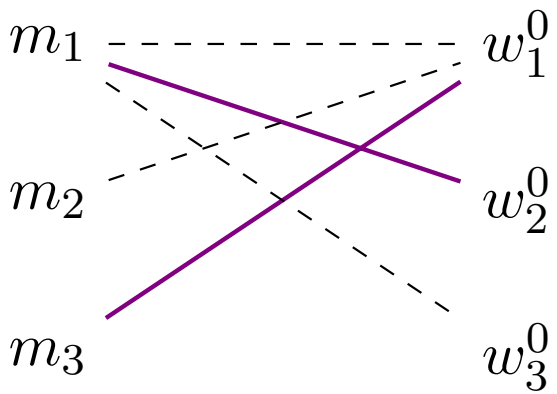

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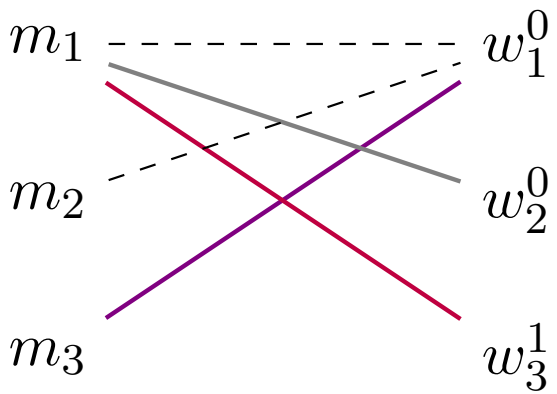

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If a vertex v had preference list $\mathbf{pref}(v)$, we give all its copies the concatenation of $\mathbf{pref}(v)^{\mu-1} \ldots \mathbf{pref}(v)^0$ padded with appropriate number of dummies, where $\mu = \mu_H$ if $v \in \mathcal{R}$ and vice-versa.

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Complexity

The new edge set is of size

$$\begin{aligned}\mu_H \mu_R \left|E\right| &\leq (\left|\mathcal{R}\right| + 2)(\sum_{h \in \mathcal{H}} q^+(h) + 2) \left|E\right| \\ &\leq (\left|\mathcal{R}\right| + 2)^2 \left|E\right| \\ &= \mathcal{O}(\left|\mathcal{R}\right|^2 \left|E\right|)\end{aligned}$$

Contents

The Students-Courses Problem

Many-to-many version of Hospital-Residents.

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Input :

- $G = (\mathcal{S} \uplus \mathcal{C}, E)$ representing students and courses, with preference lists $(E(v), <_v)$ for each vertex v.
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An edge $(s, c) \in E \setminus M$ blocks M when

- $|M(s)| < q^+(s)$ or $\exists c' \in M(s), c' <_s c$, and
- $|M(c)| < q^+(c)$ or $\exists s' \in M(c), s' <_c s$

Stability and Popularity

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- [Brandl and Kavitha, 2017] give an algorithm for maximum-cardinality popularity, similar to the algorithm in [Huang and Kavitha, 2011].

Students-Courses under Lower Quotas

One side, say students, have lower quotas $q^- : \mathcal{S} \to \mathbb{N}$

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This runs in time

$$\begin{aligned}\mu\,|E| &= (\sum_{s\in\mathcal{S}} q^-(s) + 2)\,|E| \\ &\leq (\sum_{c\in\mathcal{C}} q^+(c) + 2)\,|E| \\ &\leq (\sum_{c\in\mathcal{C}} |E(c)| + 2)\,|E| \\ &= \mathcal{O}(|E|^2)\end{aligned}$$

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- All our class upper quotas are 1.
- All courses impose the same classes.

We utilise the first two properties to simplify Huang's algorithm to our purpose.

Stability under Classifications

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$C_h^r \in \mathscr{C}_h$ be the class in which h puts r. A feasible blocking pair is a blocking pair (h, r) such that either $|M(h) \cap C_h^r| < q_h^+(C_h^r)$, or $\exists r' \in M(h) \cap C_h^r, r' <_h r$.

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Definition

M is class-stable if it has no feasible blocking pairs.

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- We adapt Huang's algorithm to a linear-time version specific to our requirement.

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Correctness : similar to Gale-Shapley.

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- We need more sophisticated constraints, like restricted edges or matroid constraints.

Questions

Thank you

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