Fault-Tolerant k-Supplier with Outliers

Deeparnab Chakrabarty, Luc Cote, **Ankita Sarkar**

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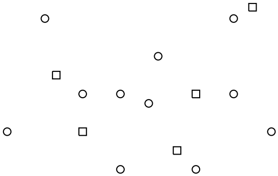
Background: k-Supplier

Input.

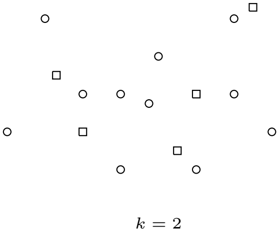
- Clients C, $|C| = n$
- Facilities F



Background: k-Supplier

Input.

- Clients C, $|C| = n$
- Facilities F
- Metric d on $C \cup F$
- Parameter k



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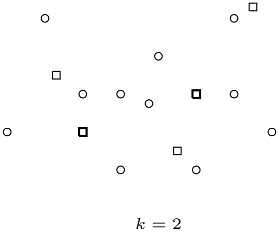
Output.

- $S \subseteq F$, $|S| \leq k$



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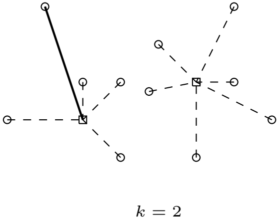
Output.

- $S \subseteq F$, $|S| \leq k$
- minimize $\max_{v \in C} d(v, S)$



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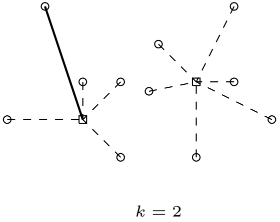

Tight 3-approximation [Hochbaum and Shmoys, 1986].

Background: Fault-tolerant k-Supplier (FkS)

Input.

- Clients C, $|C| = n$
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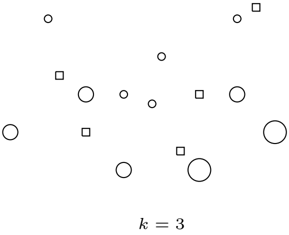
- Metric d on $C \cup F$
- Parameter k
- Fault-tolerances $\{\ell_v\}_{v \in C}$



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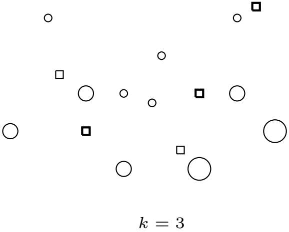
- $S \subseteq F$, $|S| \le k$
- minimize

$$\max_{v \in C} \underbrace{d_{\ell_v}(v, S)}_{\text{distance of } v \text{ to } {\ell_v}^{\text{th}} \text{ open facility}}$$



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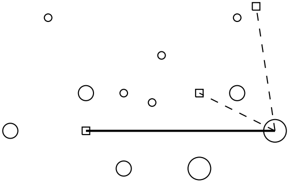

$k = 3$

Background: Fault-tolerant k-Supplier (FkS)

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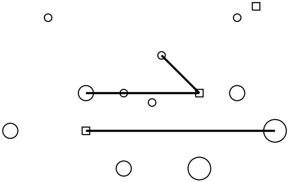
- Facilities F
- Metric d on $C \cup F$
- Parameter k
- Fault-tolerances $\{\ell_v\}_{v \in C}$



Output.

- $S \subseteq F$, $|S| \leq k$
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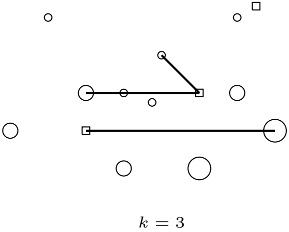
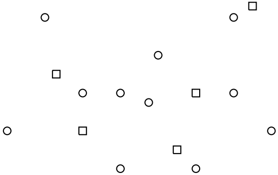


Tight 3-approximation [modified Hochbaum-Shmoys].

Background: k-Supplier with Outliers (kSO)

Input.

- Clients C, $|C| = n$
- Facilities F
- Metric d on $C \cup F$
- Parameter k
- Parameter m



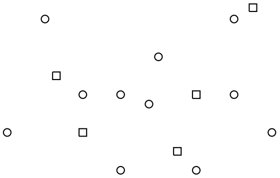
Background: k-Supplier with Outliers (kSO)

Input.

- Clients C, $|C| = n$
- Facilities F
- Metric d on $C \cup F$
- Parameter k
- Parameter m

Output.

- $S \subseteq F$, $|S| \leq k$
- $\underbrace{T}_{\text{inliers}} \subseteq C$, $|T| \geq m$,
  minimize $\max_{v \in T} d(v, S)$



$k = 2, m = 6$

Background: k-Supplier with Outliers (kSO)

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- Parameter m

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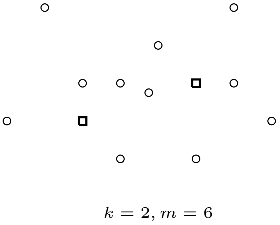

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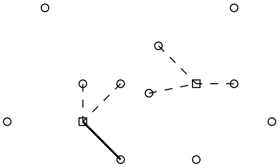

$k = 2, m = 6$

Background: kSO

Tight 3-approximation [Chakrabarty, Goyal, Krishnaswamy, 2016]

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LP-guided modified Hochbaum-Shmoys

Background: kSO

Tight 3-approximation [Chakrabarty, Goyal, Krishnaswamy, 2016]

LP-guided modified Hochbaum-Shmoys

$r := \texttt{opt}; \{\mathsf{cov}_v := \mathbf{1}_{v \in T}\}_{v \in C}$, $\{x_i := \mathbf{1}_{i \in S}\}_{i \in F}$

- $\sum_{v \in C} \mathsf{cov}_v \geq m$, $\sum_{i \in F} x_i \leq k$
- $\forall v \in C$, $\sum_{i \in B(v,r) \cap F} x_i \geq \mathsf{cov}_v$

Fault-tolerant k-Supplier with Outliers (FkSO)

Input.

- Clients C, $|C| = n$
- Facilities F
- Metric d on $C \cup F$
- Parameter k
- Parameter m
- Fault-tolerances $\{\ell_v\}_{v \in C}$

Output.

- $S \subseteq F$, $|S| \leq k$
- $\underbrace{T}_{\text{inliers}} \subseteq C$, $|T| \geq m$, minimize $\max_{v \in T} d_{\ell_v}(v, S)$

This work: approximability of FkSO

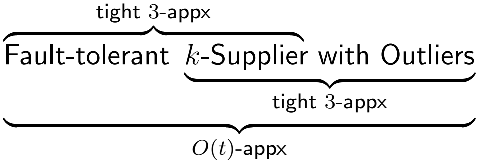
tight 3-appx

Fault-tolerant k-Supplier with Outliers

tight 3-appx

??

This work: approximability of FkSO



Main result

Where $t = |\ell_v : v \in C|$, FkSO admits a $(4t-1)$-approximation.

This work: approximability of FkSO

tight 3-appx

Fault-tolerant k-Supplier with Outliers

tight 3-appx

$O(t)$-appx

Main result

Where $t = |\ell_v : v \in C|$, FkSO admits a $(4t-1)$-approximation.

- "Uniform case" i.e. $t = 1$ admits 3-appx
- "Hochbaum-Shmoys-like algorithms" are $\Omega(t)$.

FkSO: Natural LP [CGK16, ...]

$\{\mathsf{cov}_v := \mathbf{1}_{v \in T}\}_{v \in C}$, $\{x_i := \mathbf{1}_{i \in S}\}_{i \in F}$

- $\sum_{v \in C} \mathsf{cov}_v \geq m$, $\sum_{i \in F} x_i \leq k$
- $\forall v \in C$, $\sum_{i \in B(v,r) \cap F} x_i \geq \ell_v \mathsf{cov}_v$

FkSO: Infinite integrality gap in natural LP

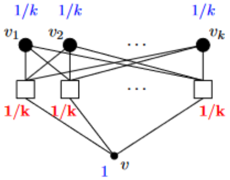
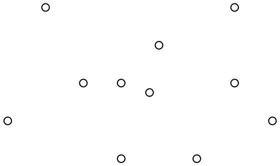


Figure 1: *One of the k identical gadgets in the gap example, showing LP values in* **red** *(x values) and blue (*cov *values). The "edges" represent distance* 1, *and all other distances are determined by making triangle inequalities tight. The fault-tolerances are* $\ell_{v_1} = \ell_{v_2} = \cdots = \ell_{v_k} = k$, *and* $\ell_v = 1$.

FkSO: Stronger LP

- Exponential-sized LP [Chakrabarty and Negahbani, 2019]
- Find solutions via round-or-cut
- cov_v's as before

Good partition: k-Supplier [HS86]



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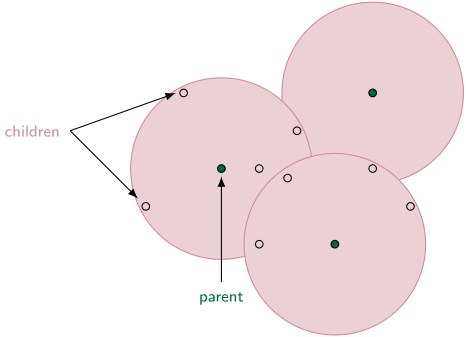

❶ parent-child structure

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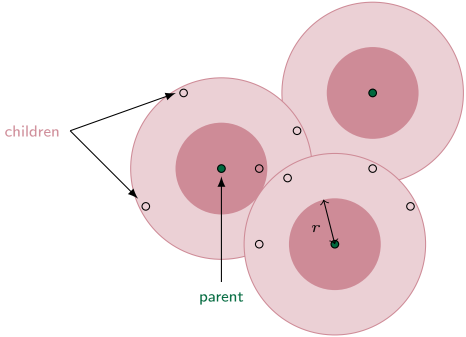

1. parent-child structure
2. parents are well-separated

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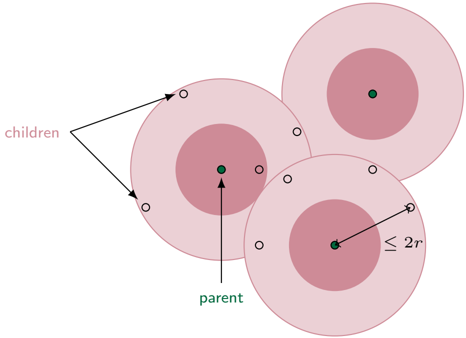

1. parent-child structure
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3. children are within $2r$ of their parents

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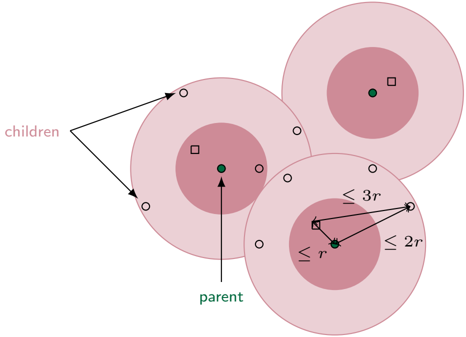

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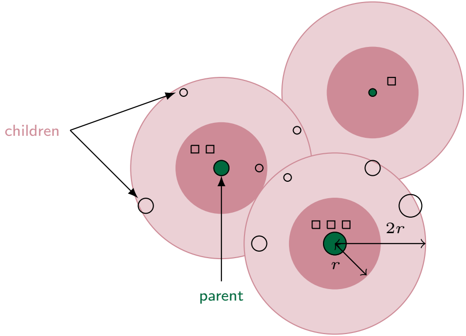
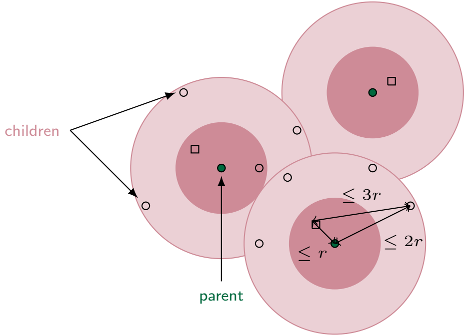


1. parent-child structure, $\ell_{\text{parent}} \geq \ell_{\text{child}}$
2. parents are well-separated
3. children are within $2r$ of their parents

Good partition: k-Supplier with Outliers [CGK16]



1. parent-child structure, $\text{cov}_{\text{parent}} \geq \text{cov}_{\text{child}}$
2. parents are well-separated
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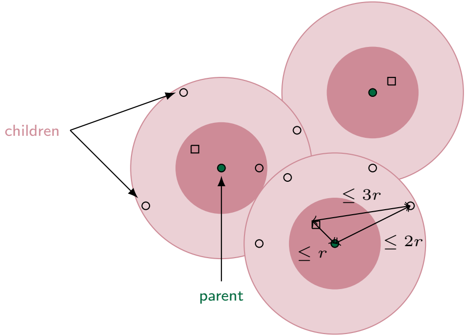
Good partition: Fault-tolerant k-Supplier with Outliers



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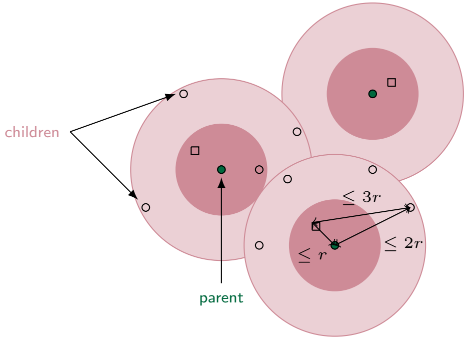


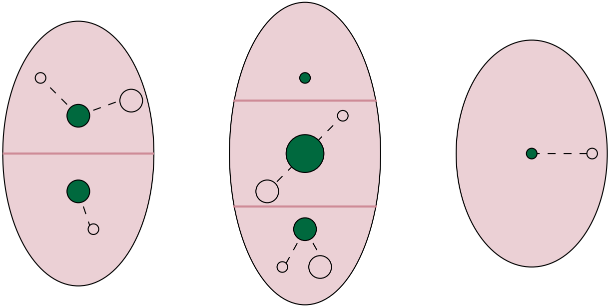
1. parent-child structure, $\ell_{\text{parent}} \geq \ell_{\text{child}}$, $\text{cov}_{\text{parent}} \geq \text{cov}_{\text{child}}$
2. ~~parents are well-separated~~
3. children are within $2r$ of their parents

ℓ_v's and cov_v's clash!

(ρ, cov)-good partition for FkSO

1. parent-child structure, $\ell_{\text{parent}} \geq \ell_{\text{child}}$, $\mathsf{cov}_{\text{parent}} \geq \mathsf{cov}_{\text{child}}$
2. well-separated **subset** of parents
3. children are within ρr of this **subset**

(ρ, cov)-good partition $\mathcal{P}$



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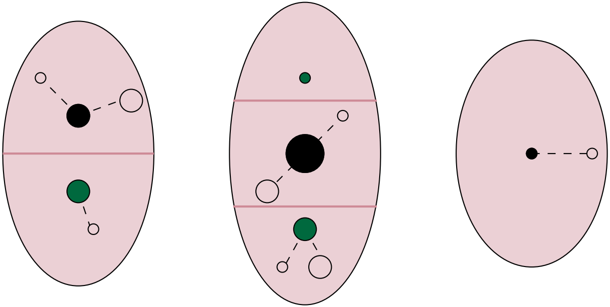

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2. $\{\textbf{grparents}(P) \in \text{parent} \cap P\}_{P \in \mathcal{P}}$ s.t. $\ell_{\textbf{grparent}} \geq \ell_{\text{parent}}$,
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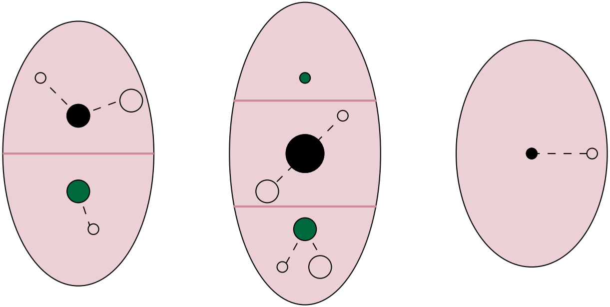

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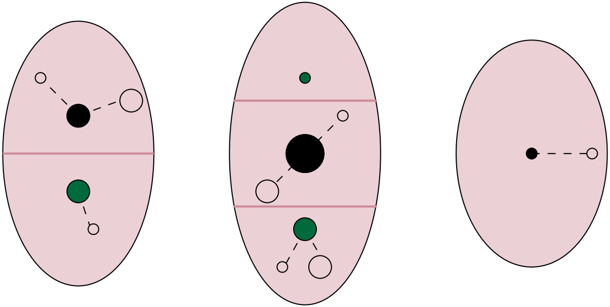
(ρ, cov)-good partition $\mathcal{P}$



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2. $\forall \{\textbf{grparents}(P) \in \text{parent} \cap P\}_{P \in \mathcal{P}}$ s.t. $\ell_{\textbf{grparent}} \geq \ell_{\text{parent}}$, **grparents** are well-separated
3. children in P are within ρr of $\textbf{grparent}(P)$

$(\rho + 1)$-approximation

Theorem (Approximation from good partition)

Given cov_v*'s from the LP, if we can construct a* (ρ, cov)*-good partition, then we can find a* $(\rho + 1)$*-approximation.*

$(\rho + 1)$-approximation

Theorem (Approximation from good partition)

Given cov_v*'s from the LP, if we can construct a* (ρ, cov)*-good partition, then we can find a* $(\rho + 1)$*-approximation.*

DP and round-or-cut.

$\rho = O(t)$

Theorem

We can construct, in polytime, a $(4t-2, \mathsf{cov})$-good partition.

$\rho = O(t)$

Theorem

We can construct, in polytime, a $(4t-2, \mathsf{cov})$*-good partition.*

Generalization of HS86, CGK16.

$\rho = \Omega(t)$

Same tension between ℓ_v's and cov_v's!

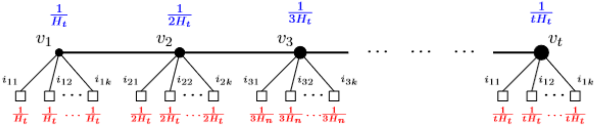

Figure 3: *An example showing the limitations of good partitions, with a solution to* (L1)-(L4) *shown in* ***red*** *(z values) and* *blue* *(*cov *values). The thin "edges" represent distance* 1*, the thick "edges" represent distance* 2*, and all other distances are determined by making triangle inequalities tight. The fault-tolerances are* $\ell_{v_1} = 1, \ell_{v_2} = 2, \ldots, \ell_{v_t} = t$.

Takeaways

- cov_v's from kSO ⚔ ℓ_v's from FkS
- (ρ, cov)-good partition $\implies$ $(\rho + 1)$-approximation
- $\rho = \Theta(t)$, where $t = \#$distinct ℓ_v's

Thank you!