Approximation Algorithms for Continuous Clustering and Facility Location Problems

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Talk at TTI Chicago
May 12, 2023

Continuous center-based k-clustering

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Input.

- Metric space (X, d). $|X|$ possibly infinite.
- Clients $C \subseteq X$, $|C| = n$
- $k \in \mathbb{N}$

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  - k-median: $\sum_{v \in C} d(v, S)$
  - k-means: $\sum_{v \in C} d(v, S)^2$

Continuous vs. discrete

- $S \subseteq X$ vs $S \subseteq C$

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  Hochbaum and Shmoys, 1985
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- Discrete best:
  - k-median: 2.613
    [Gowda, Pensyl, Srinivasan, and Trinh, SODA 2023]
  - k-means: 9
    [Kanungo, Mount, Netanyahu, Piatko, Silverman, and Wu, ComGeo 2004] ...

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- Reduction continuous $\rightarrow$ discrete

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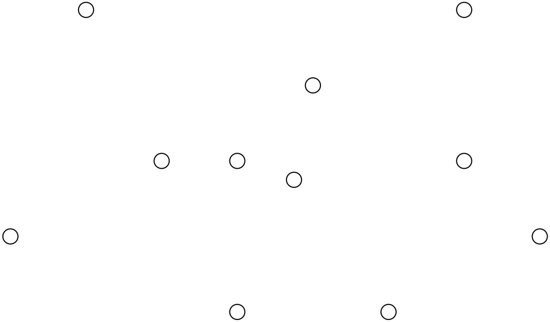

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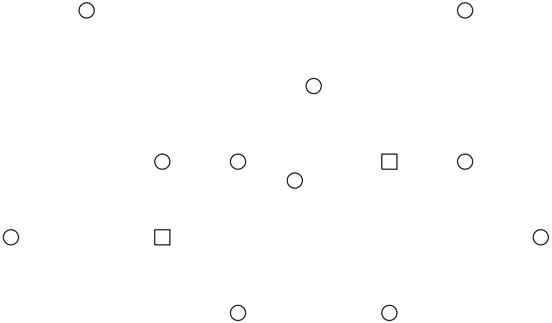

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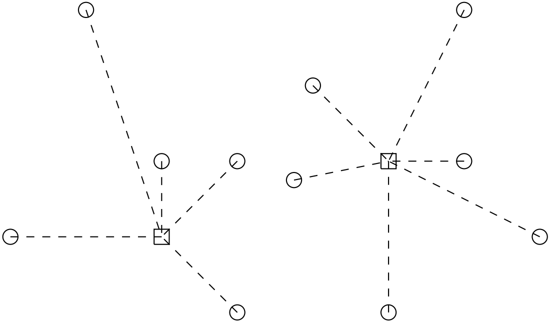

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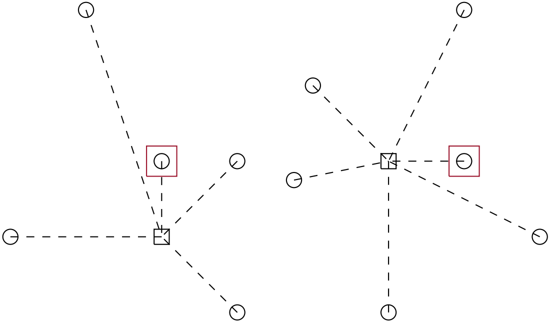

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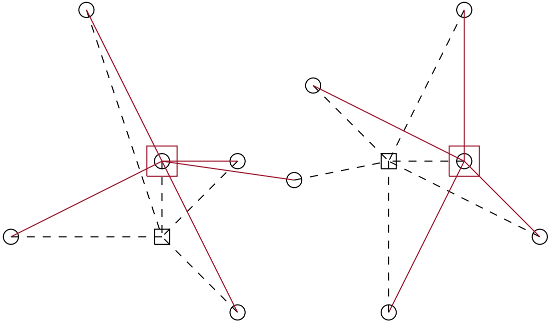

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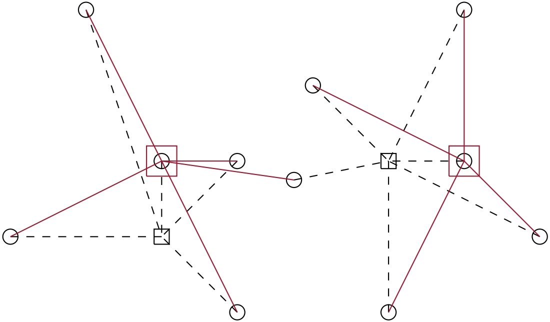

k-median: $2\times$

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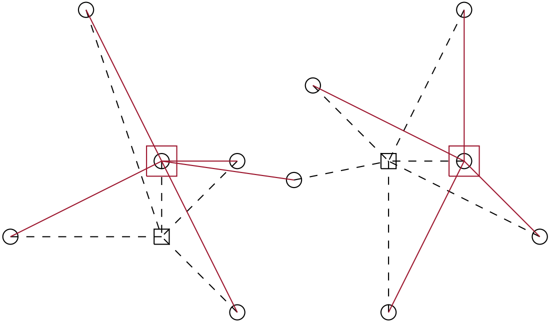

k-median: $2\times$
k-means: $4\times$

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- α-approx for discrete $\implies$ $\beta \cdot \alpha$-approx for continuous.
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- General (X, d)?

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- Fair-k-median, k-center with outliers : **Yes**.

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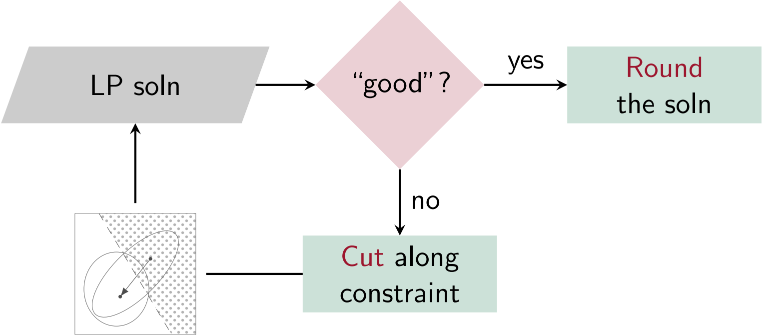

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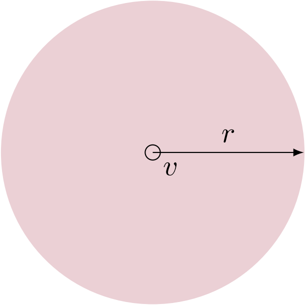

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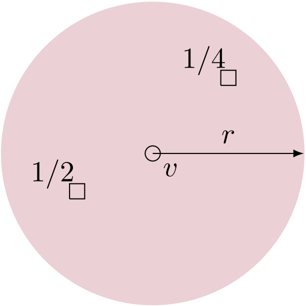

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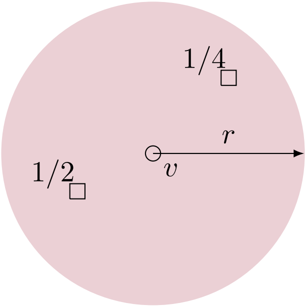

$y(v, r) = 3/4$

New LP: Consistency constraints

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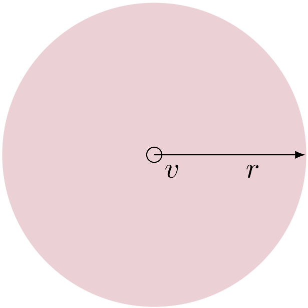

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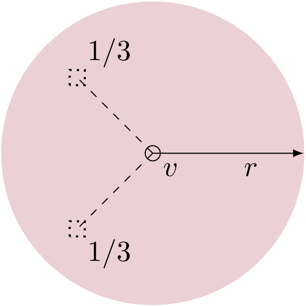

$$\underbrace{y(v, r)}_{2/3}$$

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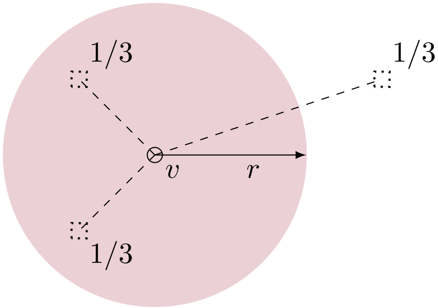

$$\underbrace{1 - y(v,r)}_{1/3}$$

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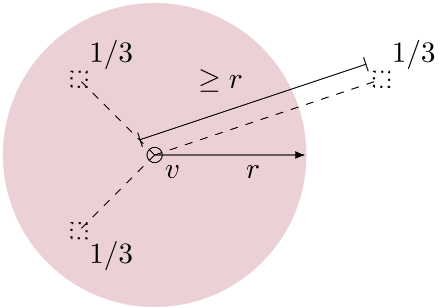

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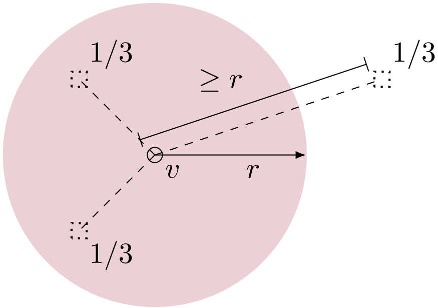

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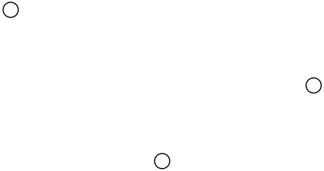

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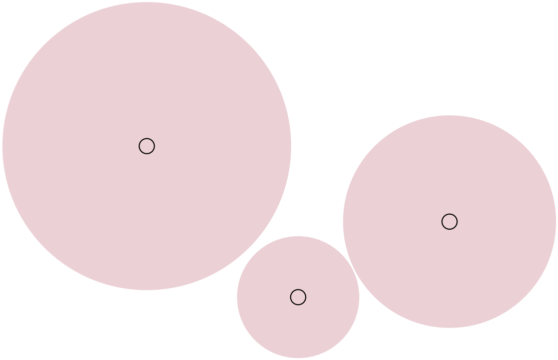

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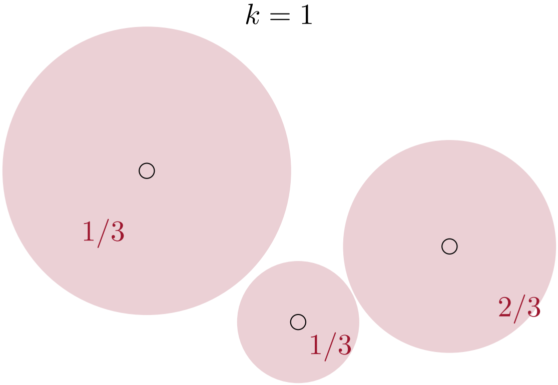

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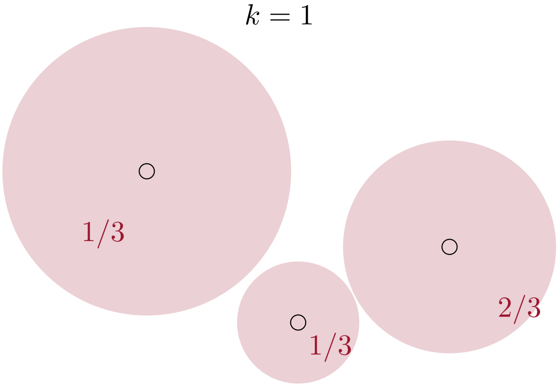

For $\mathcal{B} \subseteq \{B(v,r)\}_{v \in C, r \in \mathbb{R}}$, pairwise disjoint

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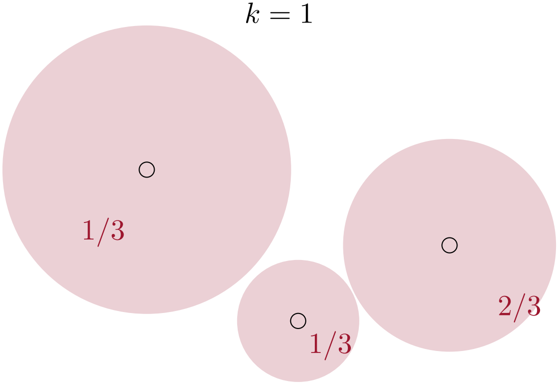
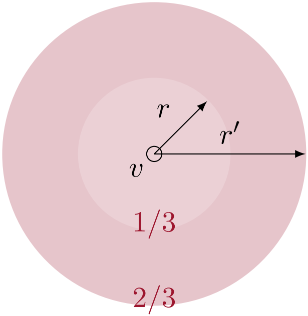


For $\mathcal{B} \subseteq \{B(v,r)\}_{v \in C, r \in \mathbb{R}}$, pairwise disjoint

$$\sum_{B \in \mathcal{B}} y(B) \leq k$$

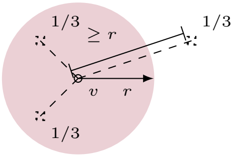
New LP: Monotonicity constraints

$y(v,r) \le y(v,r'),\ \forall v \in C,\ r \le r' \in \mathbb{R}$

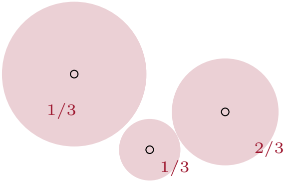


New LP: Constraints

Consistency. $\forall v \in C, r \in \mathbb{R}$,
$$C_v \geq r(1 - y(v, r))$$



Disjoint balls. $\forall$ pairwise disj. $\mathcal{B}$,
$$\sum_{B \in \mathcal{B}} y(B) \leq k$$



Monotonicity. $\forall v \in C, r \leq r' \in \mathbb{R}$,
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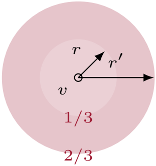
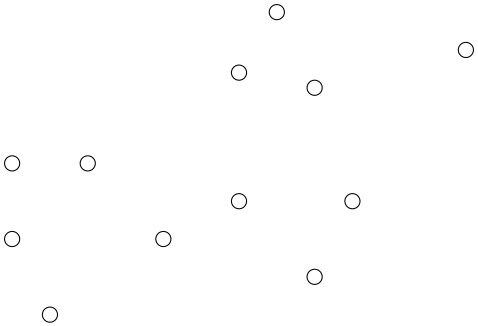


Satisfying disjoint balls constraints: filtering



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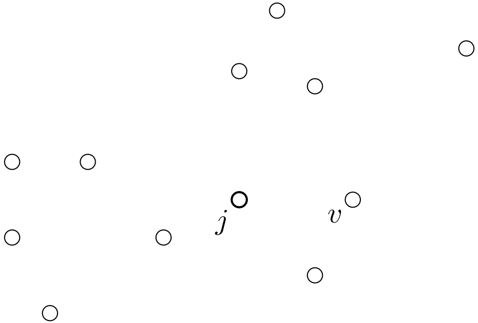

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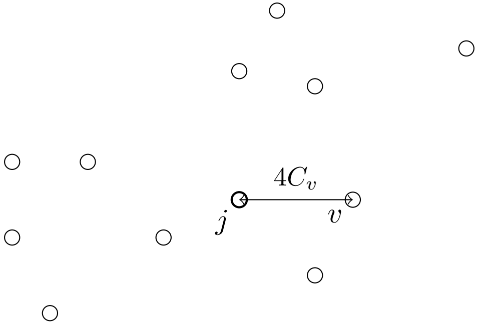

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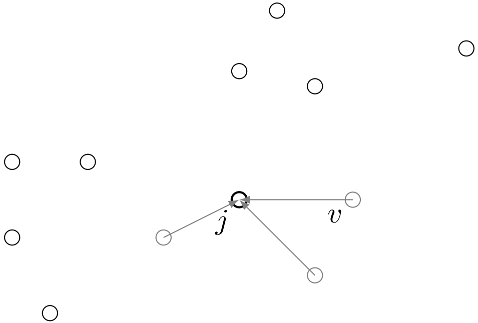

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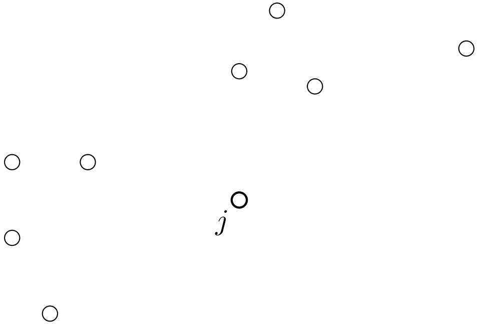

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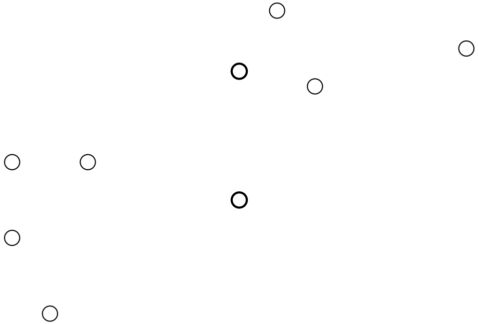

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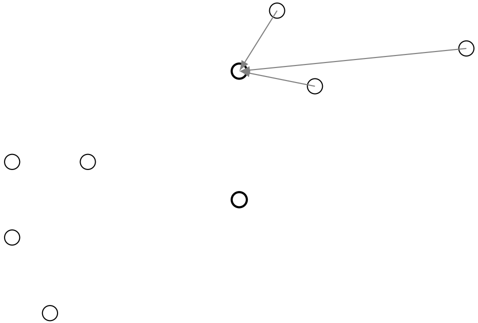

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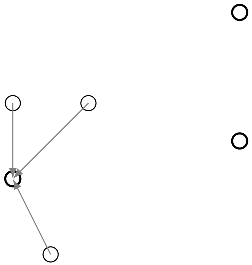

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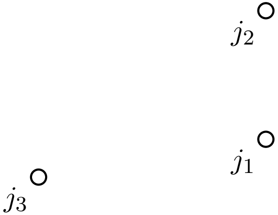

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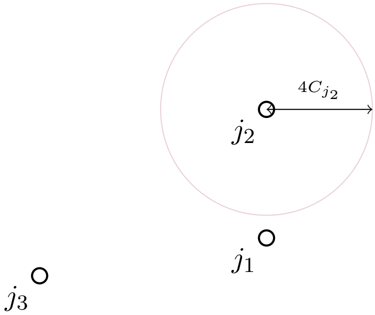

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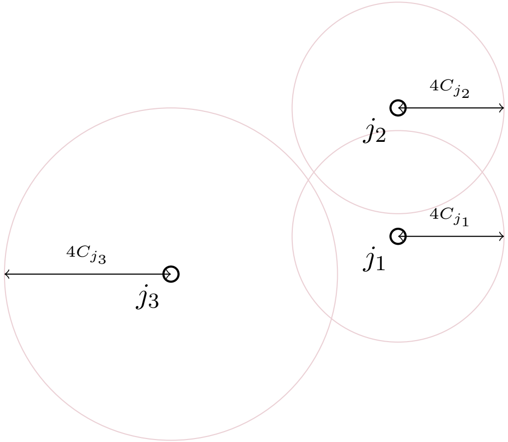

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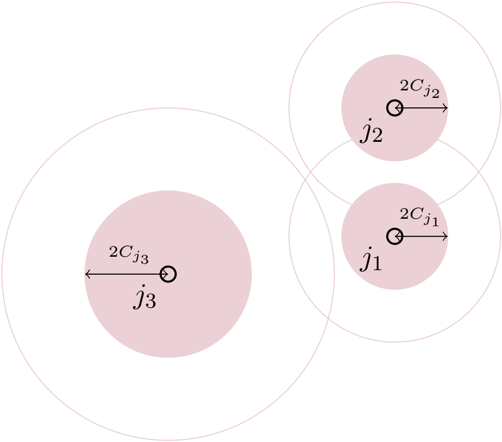

Satisfying disjoint balls constraints: round-or-cut

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(C, y)

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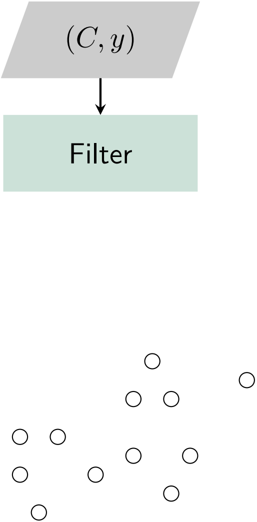

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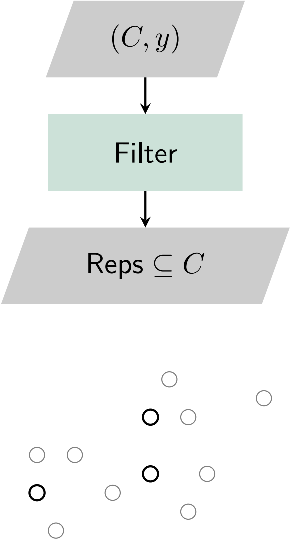

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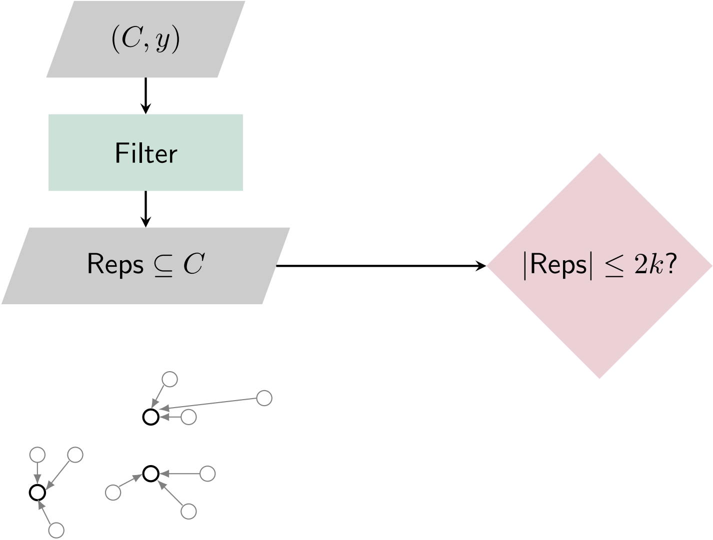

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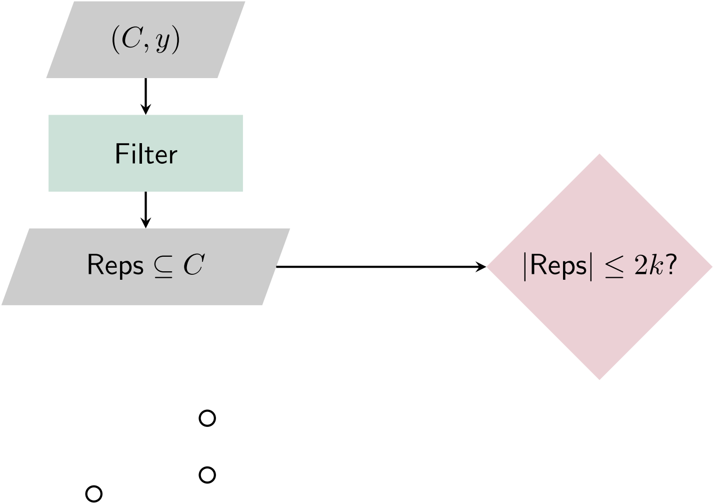

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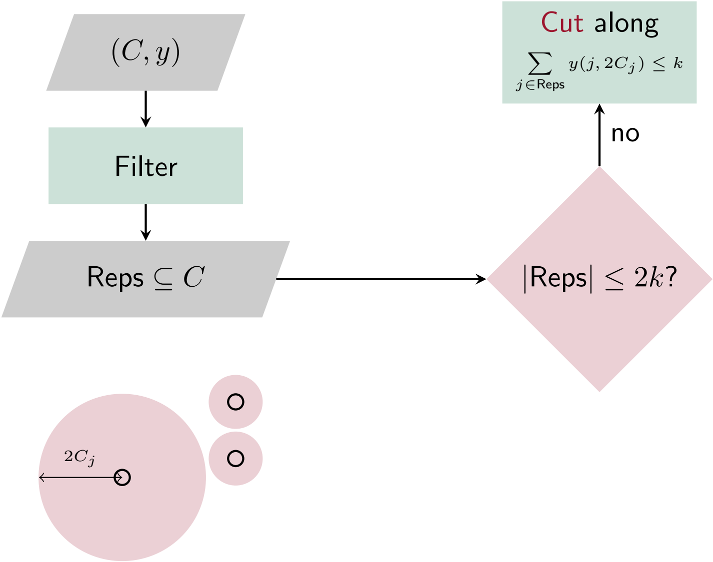

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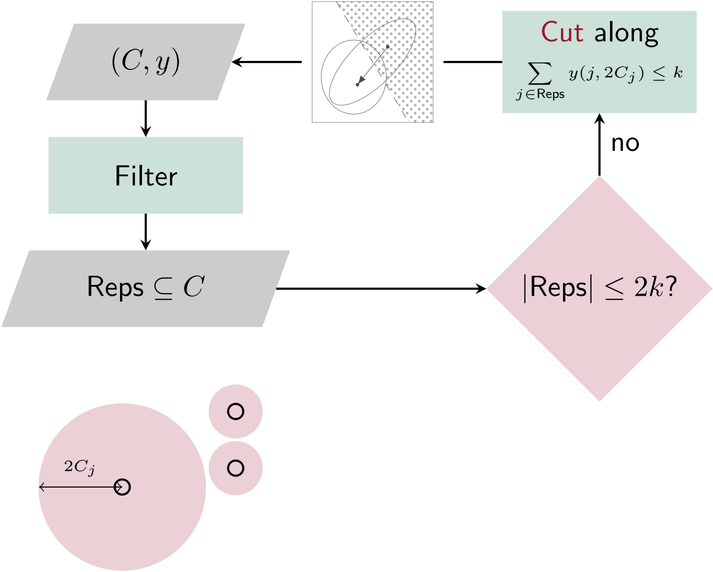

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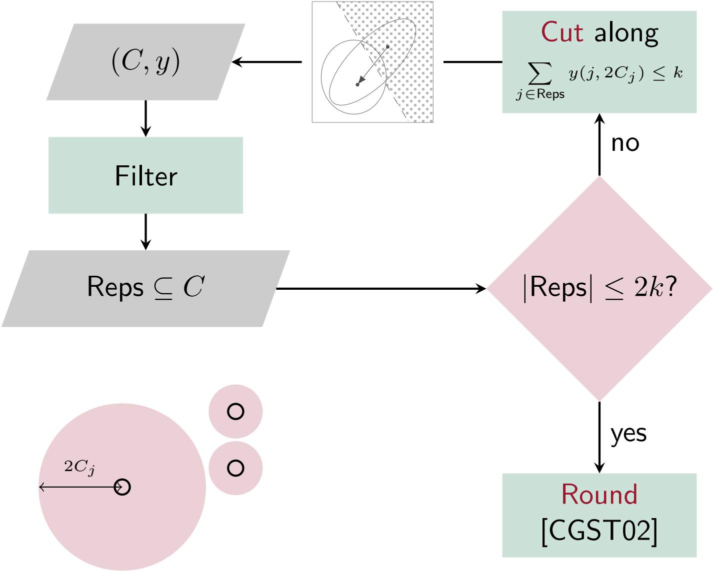
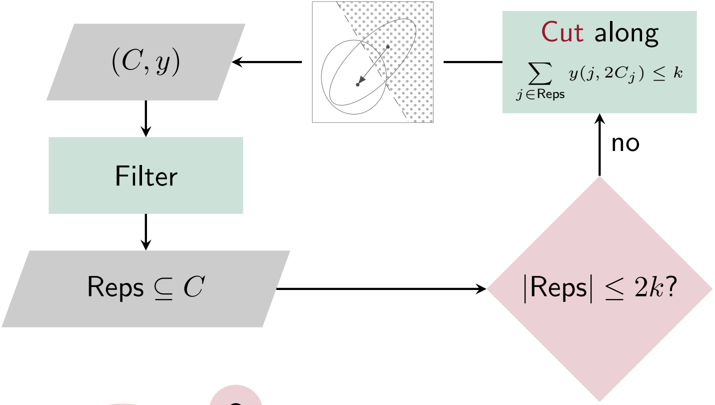
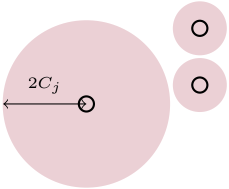


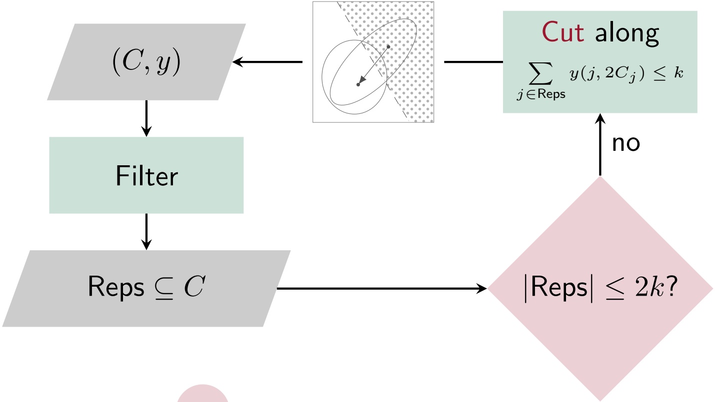
Satisfying disjoint balls constraints: round-or-cut

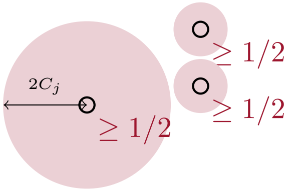




$$C_j \geq 2C_j(1 - y(j, 2C_j))$$

Satisfying disjoint balls constraints: round-or-cut





$$C_j \geq 2C_j(1 - y(j, 2C_j))$$
$$\implies \quad y(j, 2C_j) \geq 1/2$$

Open: continuous vs discrete for k-median

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  [Gowda, Pensyl, Srinivasan, and Trinh, SODA 2023]

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- $2 \times 2.613 = 5.226$
- Ours: 6.67

Question

Better than 5.226-approx for continuous k-median?

Next: Incorporating fairness

Continuous fair k-Median

Input.

- Metric space (X, d). $|X|$ possibly infinite.
- Clients $C \subseteq X$, $|C| = n$
- $k \in \mathbb{N}$
- Fairness radii $\{r_v\}_{v \in C}$

Continuous fair k-Median

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- Clients $C \subseteq X$, $|C| = n$
- $k \in \mathbb{N}$
- Fairness radii $\{r_v\}_{v \in C}$

Output.

- Facilities $S \subseteq X$, $|S| = k$
- $\forall v \in C$, $d(v, S) \leq r_v$
- Minimize $\sum_{v \in C} d(v, S)$

Continuous fair k-Median

- α-approximation: S s.t. $\sum_{v \in C} d(v, S) \leq \alpha \cdot \texttt{opt}$

Continuous fair k-Median

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- r_v's feasible? NP-hard.
- S s.t. $\forall v \in C$, $d(v, S) \leq \gamma \cdot r_v$

Bicriterion (α, γ)-approximation

- $S \subseteq X$, $|S| = k$
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Continuous vs discrete

(α, γ) for discrete $\implies$ $(2\alpha, 2\gamma)$ for continuous

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(α, γ) for discrete $\implies$ $(2\alpha, 2\gamma)$ for continuous
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Our result

There is an $(8, 8)$-approximation for continuous fair k-median.

Algorithm: the LP

- Consistency constraints
- Disjoint-balls constraints

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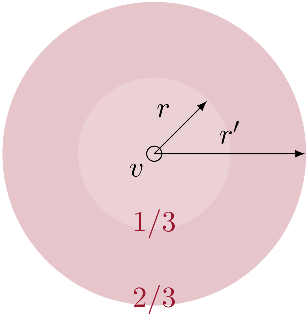
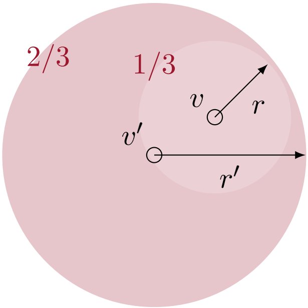


$$y(v, r) \leq y(v, r')$$

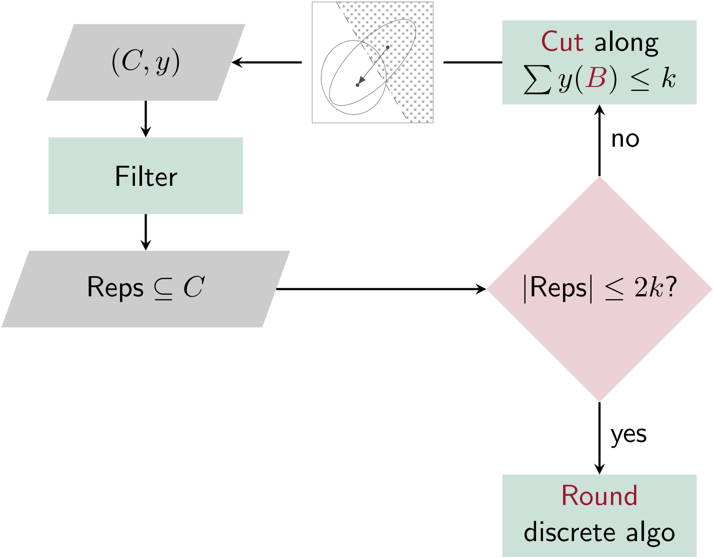
Algorithm: the LP

- Consistency constraints
- Disjoint-balls constraints
- Fairness constraints $y(v, r_v) \geq 1$
- Non-concentric monotonicity constraints



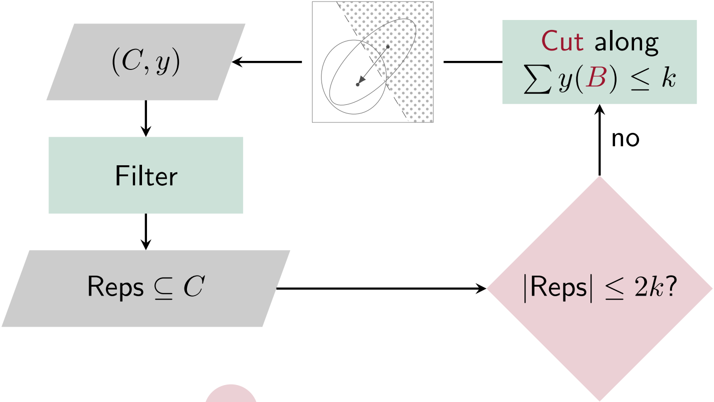
$$B(v, r) \subseteq B(v', r') \implies y(v, r) \leq y(v', r')$$

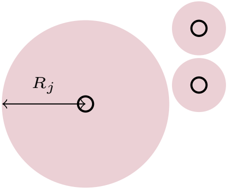
Round-or-cut



(C, y)
Filter
Reps $\subseteq C$
$|\text{Reps}| \leq 2k$?
no
Cut along
$\sum y(B) \leq k$
yes
Round
discrete algo

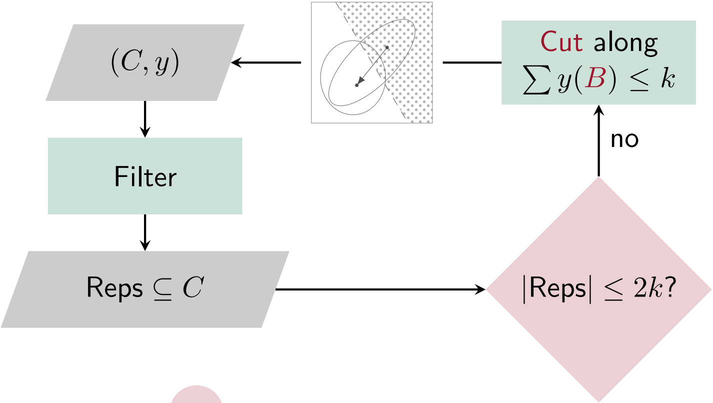
Round-or-cut

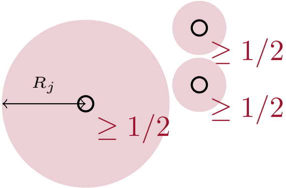




$$R_j := \min(r_j, 2C_j)$$

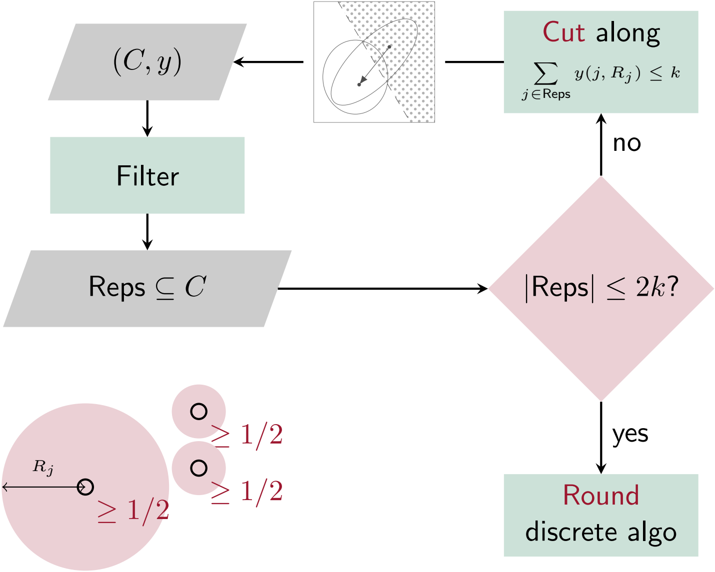
Round-or-cut





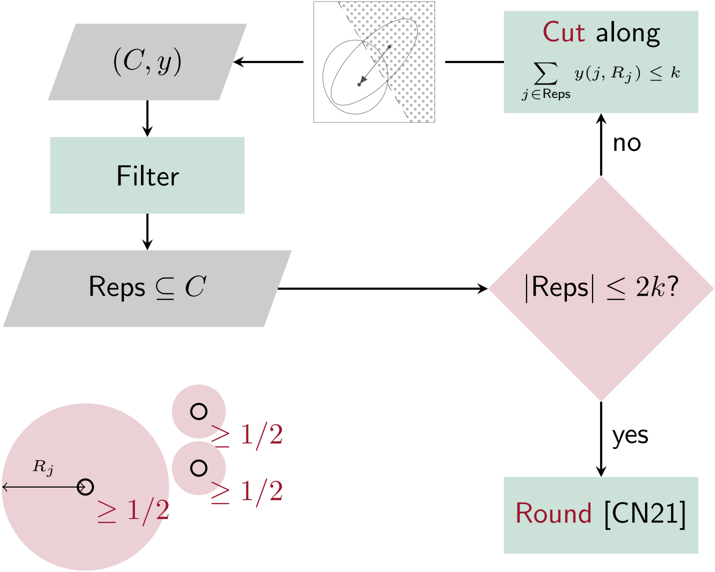
$$R_j := \min(r_j, 2C_j)$$
$$y(j, 2C_j) \geq 1/2,\ y(j, r_j) \geq 1$$

Round-or-cut



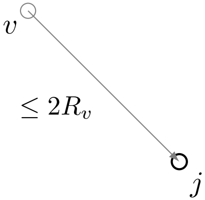
(C, y)
Cut along
$\sum_{j \in \text{Reps}} y(j, R_j) \leq k$
Filter
no
Reps $\subseteq C$
$|\text{Reps}| \leq 2k$?
yes
$\geq 1/2$
$\geq 1/2$
R_j
$\geq 1/2$
Round
discrete algo

Round-or-cut



(C, y)
Cut along
$\sum_{j \in \text{Reps}} y(j, R_j) \le k$
Filter
no
Reps $\subseteq C$
$|\text{Reps}| \le 2k$?
yes
Round [CN21]
R_j
$\ge 1/2$
$\ge 1/2$
$\ge 1/2$

Fairness guarantee [CN21]



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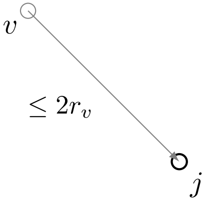

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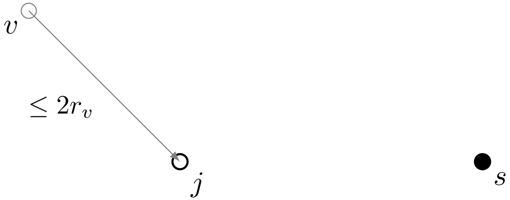

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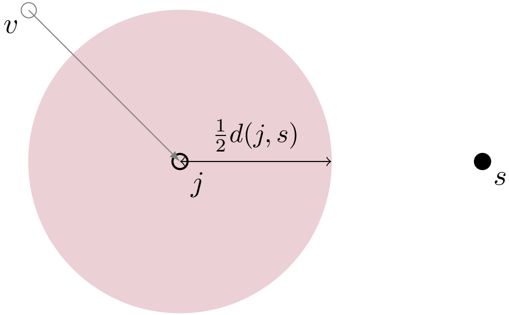
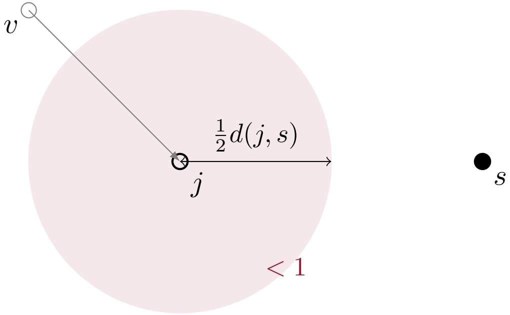
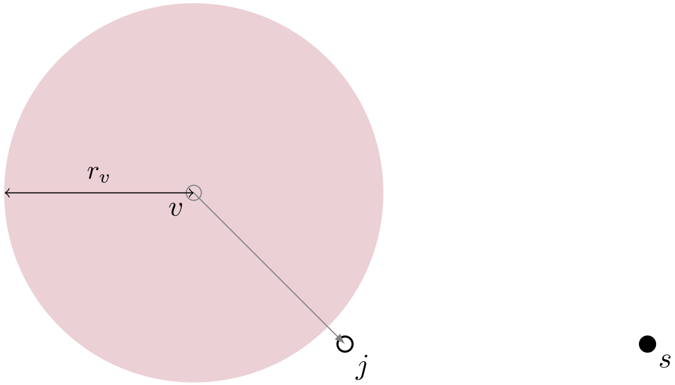


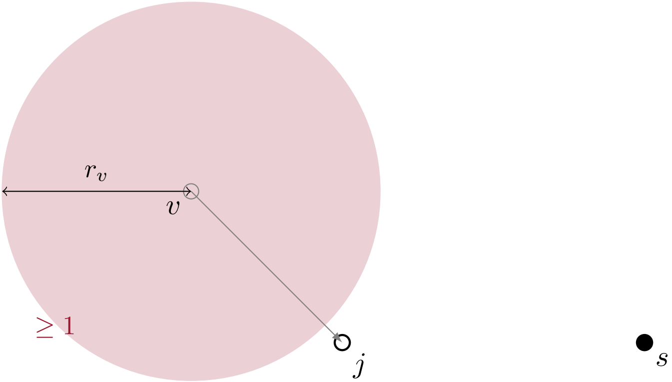
Fairness guarantee [CN21]



Fairness guarantee [CN21]



Fairness guarantee [CN21]



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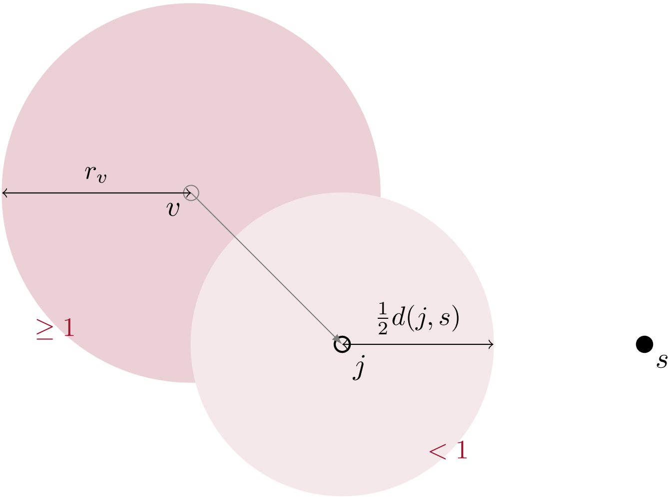

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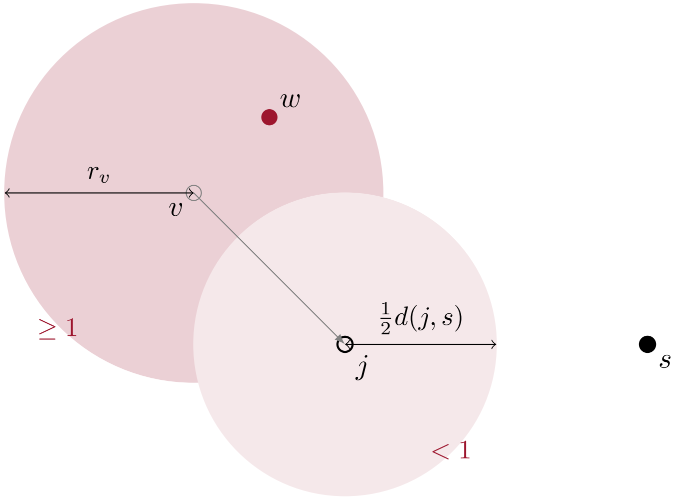
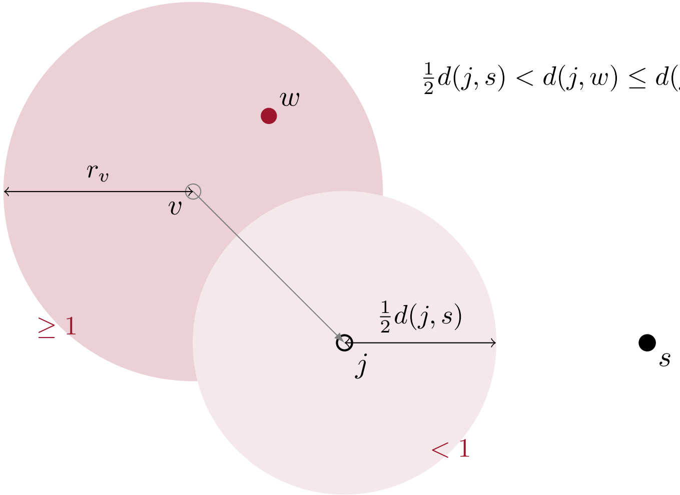


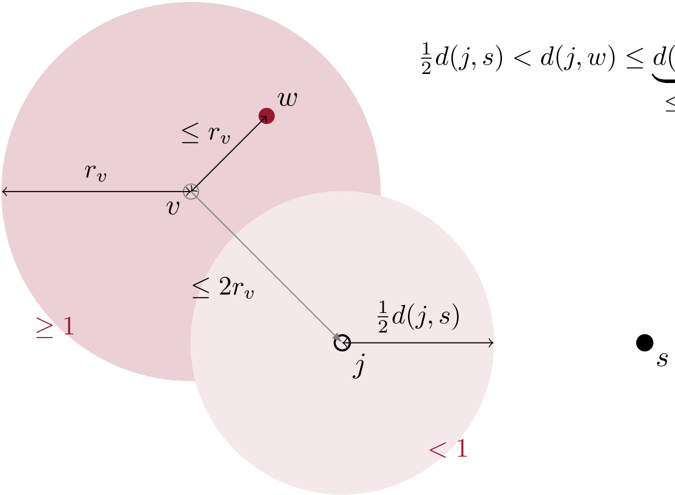
Fairness guarantee [CN21]

$$\frac{1}{2}d(j,s) < d(j,w) \leq d(j,v) + d(v,w)$$



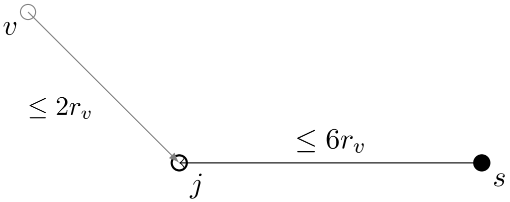
Fairness guarantee [CN21]

$$\tfrac{1}{2}d(j,s) < d(j,w) \leq \underbrace{d(j,v)}_{\leq 2r_v} + \underbrace{d(v,w)}_{\leq r_v}$$



Fairness guarantee [CN21]

$$d(j, s) \leq 6r_v \implies d(v, s) \leq 8r_v$$



Summary

- Continuous vs discrete: β gap?
- New LP and round-or-cut
- Open: continuous k-median (6.67 vs 5.226)
- Continuous fair k-median: $(8, 8)$-approximation