COMB Project : The Tutte Polynomial

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Abstract

We survey how the Tutte polynomial is defined for hyperplane arrangements and how this definition naturally relates to the Tutte polynomial for graphs. We also include a probabilistic interpretation of the Tutte polynomial of a hyperplane arrangement, and some interesting computations.

1 Introduction

From lecture notes in [2] we are familiar with a hyperplane arrangement $\mathcal{A}$ and its characteristic polynomial $\chi_{\mathcal{A}}(q)$. In this report we define the Tutte Polynomial $T_{\mathcal{A}}(x, y)$, and a transformation of it called the coboundary polynomial $\overline{\chi}_{\mathcal{A}}(q, t)$. For a graph G, definitions of Tutte polynomial $T_G(x, y)$ and coboundary polynomial $\overline{\chi}_G(q, t)$ exist, and a rank function is also defined on subsets of the edges. The bijection given in [2] that takes a graph G to $\mathcal{A}_G$, a subarrangement of the braid arrangement in $\mathbb{R}^n$ (a *graphical* arrangement), identifies edges with hyperplanes, hence suggesting that the two rank functions are "equal". We formalise this. Reconciling the definitions of Tutte polynomial for graphs and arrangements allows us to recover known results in graph theory, regarding colourings and deletion-contraction.

The recurrence relation for the Tutte polynomial for hyperplane arrangements can, in fact, be extended to non-graphical arrangements. Moreover, the coboundary polynomial of *any* hyperplane arrangement can be interpreted as an expectation on $\chi_{\mathcal{B}}(q)$ where $\mathcal{B} \subseteq \mathcal{A}$ is obtained by removing hyperplanes with independent and identical probability.

We also write the coboundary polynomial of $\mathcal{E}_n$, a generalisation of the braid arrangement, in terms of ranks of a certain class of graphs.

2 Definitions and examples

2.1 For hyperplane arrangements

$r_{\mathcal{A}}$ is the rank of a hyperplane arrangement $\mathcal{A}$.

Definition 2.1. *The* Tutte Polynomial *of a hyperplane arrangement $\mathcal{A}$ is :*

$$T_{\mathcal{A}}(x, y) = \sum_{\substack{\mathcal{A} \subseteq \mathcal{B} \\ \text{central}}} (x-1)^{r-r_{\mathcal{B}}} (y-1)^{|\mathcal{B}|-r_{\mathcal{B}}} \tag{1}$$

where $r = r_{\mathcal{A}}$ and the sum varies over all central subarrangements of $\mathcal{A}$.

Some transformations are used to define another polynomial, equivalent to the Tutte polynomial but simpler for our computations:

Definition 2.2. *The* coboundary polynomial *of a hyperplane arrangement* $\mathcal{A}$ *is :*

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{A}\subseteq\mathcal{B} \\ \text{central}}} q^{r-r_{\mathcal{B}}}(t-1)^{|\mathcal{B}|} \tag{2}$$

By computing for each summand we can verify that :

$$\overline{\chi}_{\mathcal{A}}(q,t) = (t-1)^r T_{\mathcal{A}}\Big(\frac{q+t-1}{t-1}, t\Big)$$

and

$$T_{\mathcal{A}}(x,y) = \frac{1}{(y-1)^r}\overline{\chi}_{\mathcal{A}}\big((x-1)(y-1), y\big)$$

We now look at two examples of Tutte and coboundary polynomials of arrangements. Note that the intersection over an empty collection is the entire space, hence the empty subarrangement is always central.

1. $\mathcal{A} = A_2 =$ Braid[1] arrangement in $\mathbb{R}^3$: $\forall i,j$ s.t. $1 \le i < j \le 3, x_i - x_j = 0$. $r = 2$. All subarrangements are central, so the following summands arise :

   - $\mathcal{B} = \mathcal{A}$. $r_{\mathcal{B}} = 2, |\mathcal{B}| = 3$. $(x-1)^0(y-1)^1$, $q^0(t-1)^3$
   - $|\mathcal{B}| = 2$. There are 3 subarrangements of this type, all symmetrical, $r_{\mathcal{B}} = 2$. Hence we get : $3(x-1)^0(y-1)^0$, $3q^0(t-1)^2$
   - $|\mathcal{B}| = 1$. Also 3 of these, symmetrical cases. We get : $3(x-1)^1(y-1)^0$, $3q^1(t-1)^1$
   - $\mathcal{B} = \emptyset$. We get $(x-1)^2(y-1)^0$, $q^2(t-1)^0$

   Thus

$$\begin{aligned} T_{\mathcal{A}}(x,y) &= (y-1) + 3 + 3(x-1) + (x-1)^2 \\ &= y - 1 + 3x + (x^2 - 2x + 1) \\ &= x^2 + x + y \end{aligned}$$

   and

$$\begin{aligned} \overline{\chi}_{\mathcal{A}}(q,t) &= (t-1)^3 + 3(t-1)^2 + 3q(t-1) + q^2 \\ &= (t-1)^2\Big((t-1) + 3 + \frac{3q}{t-1} + \frac{q^2}{(t-1)^2}\Big) \\ &= (t-1)^2\Big(\frac{q^2 + (t-1)^2 + 2q(t-1)}{(t-1)^2} + \frac{q+(t-1)}{t-1} + t + 1 - 1\Big) \\ &= (t-1)^2 T_{\mathcal{A}}\Big(\frac{q+t-1}{t-1}, t\Big) \end{aligned}$$

[1] Recall : the braid arrangement in $\mathbb{R}^n$ is the Coxeter arrangement of type A and rank $n-1$

as desired. Also

$$\begin{aligned}\frac{1}{(y-1)^2}\overline{\chi}_{\mathcal{A}}\big((x-1)(y-1),y\big) &= (y-1)+3+3(x-1)+(x-1)^2\\ &= (x-1+1)^2+(x-1)+2+(y-1)\\ &= T_{\mathcal{A}}(x,y)\end{aligned}$$

2. $\mathcal{A}$ = Linial arrangement in $\mathbb{R}^3$: $\forall i,j$ s.t. $1 \le i < j \le 3, x_i - x_j = 1$. So $r = 2$. All subarrangements other than $\mathcal{A}$ itself are central. Hence, summing for subsets of sizes $2, 1, 0$ respectively :

$$\begin{aligned}T_{\mathcal{A}}(x,y) &= 3(x-1)^0(y-1)^0+3(x-1)^1(y-1)^0+(x-1)^2(y-1)^0\\ &= 3+3(x-1)+(x-1)^2\\ &= (x-1+1)^2+(x-1)+2\\ &= x^2+x+1\end{aligned}$$

2.2 For graphs

By a "graph", we mean a simple undirected graph, i.e. $G = (V, E)$ where E is an irreflexive symmetric binary relation on the elements of V. We shall assume $V = \big[|V|\big]$. Due to symmetry of E i.e. undirected edges, we shall wlog denote an edge $(i, j) = (j, i)$ by ij where $i < j$.
K_n, P_n, C_n are respectively the complete graph, the path graph, and the cycle graph on n vertices.

Definition 2.3. *In $G = (V, E)$, for a subset of edges $A \subseteq E$, $k(A)$ is the number of connected components in $G|_A = (V, A)$. $k(E)$ is denoted c, the number of connected components in G.*

Definition 2.4. *The* Tutte polynomial[2] *of a graph $G = (V, E)$ is*

$$T_G(x,y) = \sum_{A\subseteq E}(x-1)^{k(A)-k(E)}(y-1)^{|A|+k(A)-|V|} \tag{3}$$

Definition 2.5. *For a subset of edges $A \subseteq E$, the* rank *in G of A is :*

$$r_G(A) = |V| - k(A)$$

and $r_G(E)$ is denoted r_G.

We immediately notice that on rewriting Definition 2.4 in terms of $r_G(A)$ instead of $k(A)$, it resembles Definition 2.1.
Imitating the transformation in case of arrangements, and replacing r by r_G, we obtain

Definition 2.6. *The* coboundary polynomial *of a graph $G = (V, E)$ is*

$$\overline{\chi}_G(q,t) = \sum_{A\subseteq E} q^{k(A)-k(E)}(t-1)^{|A|} \tag{4}$$

[2]There are a number of equivalent definitions, e.g. Tutte's original definition in terms of spanning trees. The definition we use is prevalent today, and the easiest to relate with hyperplane arrangements.

hence in this case also computing the Tutte and coboundary polynomials are equivalent.
Let us now look at an example:
$G = K_3 = ([3], 2^{[3]})$. Hence $c = 1, r_G = 2$. The summands are :

- $A = E$. We get $(x-1)^0(y-1)^1$
- $|A| = 2$. We have 3 symmetric cases. Hence we get $3(x-1)^0(y-1)^0$, since (V, A) is still connected.
- $|A| = 1$. Again we have 3 symmetric cases. In this case $k(A) = 2$. Hence we get $3(x-1)^1(y-1)^0$
- $|A| = 0$. $k(\emptyset) = |V| = 3$. We get $(x-1)^2(y-1)^0$

We know that $A_2 = \mathcal{A}_{K_3}$. It is apparent that the summands above correspond exactly to those of T_{A_2} s.t. the summand for each $A \subseteq E$ equals that for $\mathcal{A}_{G|_A}$; so of course $T_G(x,y) = T_{A_2}(x,y) = T_{\mathcal{A}_G}(x,y) = x^2 + x + y$. But moreover, $r_{K_3} = r_{A_2}$, and the Tutte polynomials are equal between all possible subgraphs and corresponding subarrangements. This motivates the next section.

3 Preservation of the Tutte Polynomial

Let us recall the bijection in [2] : a graph G on n vertices is associated with an arrangement $\mathcal{A}_G \subseteq A_{n-1} \subseteq 2^{\mathbb{R}^n}$, s.t. $\forall i < j$ s.t. $i, j \in [n]$, $ij \in E(G) \Leftrightarrow H_{ij} \in \mathcal{A}_G$, where $H_{ij} \equiv x_i - x_j = 0$. The main result of this section is :

Theorem 3.1. *Given a graph G, $T_G(x,y) = T_{\mathcal{A}_G}(x,y)$*

The proof, outlined in [1], is via the following ideas:

- Rewriting Definition 2.4 in terms of $r_G(A), r_G$ we get :

$$\begin{aligned} T_G(x,y) &= \sum_{A \subseteq E} (x-1)^{(|V|-r_G(A))-(|V|-r_G)}(y-1)^{|A|+(|V|-r_G(A))-|V|} \\ &= \sum_{A \subseteq E} (x-1)^{r_G - r_G(A)}(y-1)^{|A|-r_G(A)} \end{aligned} \tag{5}$$

- The obvious bijection between 2^E and (central, but in this case all) subarrangements of $\mathcal{A}_G$
- Lemma 3.1

Lemma 3.1. *Given a graph G, $r_G = r_{\mathcal{A}_G}$.*

We shall state and outline the proof of a stronger lemma which shall imply Lemma 3.1. The full force of the stronger lemma shall be required later.

Definition 3.1. *A graded graph $G = (V, E, h)$ is a graph (V, E) with a function $h : V \to \mathbb{N}$, called a* grading *on (V, E). For a vertex $v \in V$, $h(v)$ is called the* height *of v. For any $r \in \mathbb{N}$, $h^{-1}(r)$ is said to be the r^{th}* level *of G.*

Lemma 3.2. *Given a graded graph $G = (V, E, h)$, define $\mathcal{A}_G$ as follows : $\mathcal{A}_G = \{x_i - x_j = h(i) - h(j)$ s.t. $1 \le i < j \le n\}, n = |V|$. Then $r_G = r_{\mathcal{A}_G}$.*

This clearly implies Lemma 3.1 since $G = (V, E)$ can be viewed as $G = (V, E, h), h : V \to \{0\}$. The definitions of $\mathcal{A}_G$ in Lemma 3.2 and [2] then reconcile on non-graded graphs.
The key ideas of the proof to Lemma 3.2 are as follows :
(Recall the notion of "independence" among hyperplanes, defined to be the linear independence of normals on them from the origin.)

- Dependence in $\mathcal{A}_G$ arises from cycles in G, with minimal dependent[3] subarrangements corresponding to cycles. Example : $x_1 = x_2, x_2 = x_3, x_1 = x_3$ are dependent, but removing any one makes them independent; they correspond to a triangle subgraph.
- F be a spanning forest of G, then $r_F = r_G$ (same connected components). Since F is a maximal acyclic subgraph, $\mathcal{A}_F$ is a maximal independent subarrangement.

4 Finite field method

We know from [2] that for a large enough prime power q, a $\mathbb{Z}$-arrangement $\mathcal{A} \subseteq 2^{\mathbb{R}^n}$ "reduces correctly" over $\mathbb{F}_q^n$. Using this, we have seen a finite field method for computing $\chi_{\mathcal{A}}(q)$. There is also a finite field method for Tutte polynomials, arising from the following result in [1]:

Theorem 4.1. *$\mathcal{A} \subseteq 2^{\mathbb{R}^n}$ be a $\mathbb{Z}$-arrangement of rank r, q a large enough prime power, and $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then :*

$$q^{n-r}\overline{\chi}_{\mathcal{A}}(q, t) = \sum_{p \in F_q^n} t^{h(p)} \tag{6}$$

where $h(p) = |H(p)|, H(p) = \{H \in \mathcal{A} \text{ s.t. } \in H \cap \mathbb{F}_q^n\}$

The proof is from [1]. It relies on the fact that $q^{dim \cap \mathcal{B}} = |\cap \mathcal{B}_q|$ for a subarragement $\mathcal{B}$.
Let us illustrate the above result on the familiar $\mathcal{A} = A_2$, with $q = 5$. Clearly the coefficient for t^i is $|p \in F_q^n \text{ s.t. } h(p) = i\}|$. Let us call these numbers s_i.

- $s_3 = 5$, since we only choose one value for all coordinates.
- $s_2 = 0$ since $x_1 = x_2, x_2 = x_3 \Rightarrow x_2 = x_3$ and so on.
- $s_1 = 3.5.4 = 60$ since we choose two coordinates to be equal, choose their value, and choose a different value for the remaining coordinate.
- $s_0 = 5^3 - (s_1 + s_2 + s_3) = 125 - 65 = 60$ i.e. all remaining points.

Then suitably substituting in LHS of Equation 6

$$\begin{aligned} 5^{3-2}\overline{\chi}_{A_2}(5, t) &= 5\big[(t-1)^3 + 3(t-1)^2 + 15(t-1) + 25\big] && \text{...from earlier example} \\ &= 5\big[t^3 - 3t^2 + 3t - 1 + 3(t^2 - 2t + 1) + 15(t-1) + 25\big] \\ &= 5(t^3 + 12t + 12) \\ &= 5t^3 + 60t + 60 \end{aligned}$$

which, from our calculations of s_i's, is the RHS of Equation 6, as desired.

[3]This idea gives the connection to Tutte Polynomial of matroids

5 Recovering known graph theoretic results

5.1 Graph colourings

Definition 5.1. *A* colouring[4] κ *of a graph* $G = (V, E)$ *is a map* $V \to \mathbb{N}$.

Definition 5.2. $\kappa : V \to \mathbb{N}$ *is a* q*-colouring of* (V, E) *if the image of* κ *is of size* $\leq q$*, i.e. "upto* q *colours are used".*

Definition 5.3. *An edge* $(u, v) \in E$ *is said to be* monochromatic *in a colouring* κ *of* (V, E) *if* $\kappa(u) = \kappa(v)$.

There is a well known result :

Theorem 5.1. *([3, Proposition 6.3.26])*

$$q^c \overline{\chi}_G(q, t) = \sum_{\substack{q\text{-colourings} \\ \kappa \text{ of } G}} t^{\text{mono}(\kappa)}$$

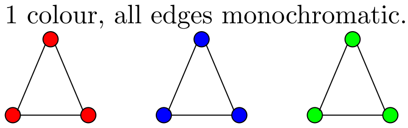
where mono(κ) *is the number of edges of* G *that are monochromatic in* κ.

Now consider Theorem 3.1 and the Lemma 3.1 used to prove it. They give LHS of Theorem 5.1 to be $q^{n-r}\overline{\chi}_{\mathcal{A}_G}(q, t)$, where $r = r_{\mathcal{A}_G}$; i.e it equals the LHS of Theorem 4.1. Now we can interpret the RHS of Theorem 4.1 to recover Theorem 5.1 : note that every $p \in \mathbb{F}_q^n$ corresponds one-to-one with a q-colouring κ_p of G, s.t. an element of $H(p)$ corresponds to a monochromatic edge.
We now see an example, with $G = K_3, \mathcal{A}_G = A_2, q = 3$. Using the computations in earlier examples:

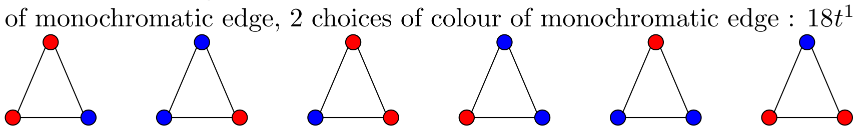
$$\begin{aligned} 3^1 \overline{\chi}_G(3, t) &= 3\big[(t-1)^3 + 3(t-1)^2 + 9(t-1) + 9\big] \\ &= 3\big[(t^3 - 3t^2 + 3t - 1) + 3(t^2 - 2t + 1) + 9(t-1) + 9\big] \\ &= 3(t^3 + 6t + 2) \\ &= 3t^3 + 18t + 6 \end{aligned}$$

We will see from the figures below (using colours red, blue, green) that the values of mono(κ) are as desired.

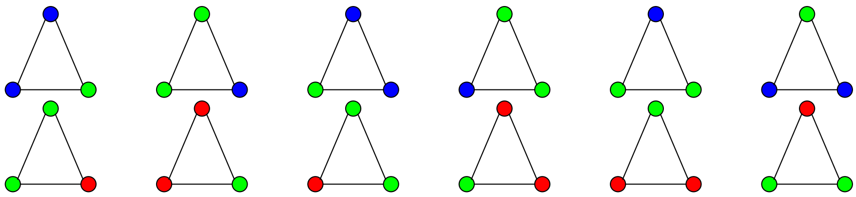
- 1 colour, all edges monochromatic. 3 choices of colour : $3t^3$.



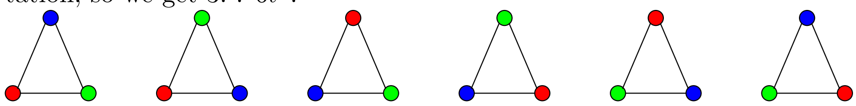
- if two edges are monochromatic, so is the third : $0t^2$
- 2 colours, one edge monochromatic. 3 choices of colour subset, 3 choices of monochromatic edge, 2 choices of colour of monochromatic edge : $18t^1$



[4]In our definition "colouring" *does not* mean "proper colouring" by default



- 3 colours, no monochromatic edges. Colour assignment is then a permutation, so we get $3! : 6t^0$.



5.2 Deletion-contraction recurrence

Recall that in a graph $G = (V, E)$ a deletion of an edge $e \in E$ gives the graph

$$G - e = (V, E - e)$$

where $E - e = E \setminus \{e\}$
and the contraction of $e = uv$ gives the graph

$$G/e = (V', E')$$

where $V' = ((V \setminus \{u, v\}) \cup \{w\},\ w \notin V$
and $E' = (E \setminus \{ij \text{ s.t. } \{u, v\} \cap \{i, j\} \neq \emptyset\}) \cup \{iw \text{ s.t. } iu \in E \text{ or } iv \in E\}$.
Also in a hyperplane arrangement $\mathcal{A}$, for a fixed hyperplane H_0, we defined the deletion of H_0:

$$\mathcal{A}' = \mathcal{A} - H_0 = \mathcal{A} \setminus \{H_0\}$$

which has the same ambient space as $\mathcal{A}$, and the restriction w.r.t. H_0 :

$$\mathcal{A}'' = \mathcal{A}/H_0 = \{H \cap H_0, H \in \mathcal{A}'\}$$

where the ambient space is H_0.
We also established in class that :

$$\mathcal{A}_{G-ij} = \mathcal{A}_G - H_{ij} \tag{7}$$

$$\mathcal{A}_{G/ij} = \mathcal{A}_G / H_{ij} \tag{8}$$

where H_{ij} is the hyperplane corresponding to ij in our usual bijection.

Definition 5.4. *In a graph $G = (V, E)$, an edge $e \in E$ is a* bridge *if $k(E-e) = k(E) + 1$ i.e. its deletion disconnects a component (or, $\{e\}$ is a cut in G).*

There is a known recursive formula for Tutte polynomial in terms of deletion-contraction of an edge that is not a loop (edge of a vertex with itself) or bridge. In the context of bijection with arrangements, loops do not make sense. With this in mind, we state the recurrence :

Theorem 5.2. *For a graph $G = (V, E)$ and an $e \in E$ s.t. e is not a bridge*

$$T_G = T_{G-e} + T_{G/e}$$

In fact, this formula is a recursive definition for T_G, equivalent to the definition we have used. If we assume this, then the similar result for arrangements, Theorem 5.3 is obviously a recursive definition for $T_{\mathcal{A}}$, at least for braid subarrangements.
However, we can prove Theorem 5.3 for *all* hyperplane arrangements without using graph theoretic results. This will immediately imply Theorem 5.2 from Equations 7 and 8, since the rank condition in Theorem 5.3 corresponds exactly to the bridge condition in Theorem 5.2.

Theorem 5.3. *For a hyperplane arrangement $\mathcal{A}$ and $H \in \mathcal{A}$ s.t. $r_{\mathcal{A}} = r_{\mathcal{A}-H}$, $T_{\mathcal{A}} = T_{\mathcal{A}-H} + T_{\mathcal{A}/H}$.*

A proof of the above is due to [1] via proving the equivalent statement $\overline{\chi}_{\mathcal{A}}(q,t) = \overline{\chi}_{\mathcal{A}-H}(q,t) + (t-1)\overline{\chi}_{\mathcal{A}/H}(q,t)$. The $(t-1)$ factor arises from the decrease in cardinality which when written as Tutte polynomial is compensated by decrease in ambient dimension. We use the following ideas:

- Multiply the equation by q^{n-r}, $r = r_{\mathcal{A}}$ – this is the expression to be actually proved.
- Using Theorem 4.1, replace LHS in above by a summation over $p \in \mathbb{F}_q^n$; split the sum over $p \in H$ and $p \notin H$ and use Theorem 4.1 on each sum.

Our favourite example is $G = K_3, \mathcal{A}_G = A_2$. However edge contraction and hyperplane restriction in this example requires us to allow parallel edges and hyperplanes with multiplicity. This arises for easy examples since braid subarrangements are central. Let us define :

Definition 5.5. [5] *A* multigraph $G = (V, E, M)$ *consists of a set V wlog $= \big[|V|\big]$, E an irreflexive symmetric binary relation on V, and $M : E \to \mathbb{N}$.*

Clearly our usual graph is where $\forall e \in E, M(e) = 1$.

Definition 5.6. *A* hyperplane arrangement with multiplicity *is a pair $(\mathcal{A}, M)$ where $\mathcal{A}$ is a hyperplane arrangement and $M : \mathcal{A} \to \mathbb{N}$. By abuse of notation we will denote it $\mathcal{A}$ when multiplicity is clear from context.*

Again, our usual hyperplane arrangement is where $\forall H \in \mathcal{A}, M(H) = 1$. The usual association $G \leftrightarrow \mathcal{A}_G$ can be extended naturally to $G = (V, E, M_G) \leftrightarrow \mathcal{A}_G = (\mathcal{A}_{(V,E)}, M_{\mathcal{A}})$ s.t. for each edge $e \in E$ and its corresponding hyperplane $H_e \in \mathcal{A}_{(V,E)}$, $M_G(e) = M_{\mathcal{A}}(H_e)$.
Now, we know that $T_{K_3}(x,y) = T_{A_2}(x,y) = x^2 + x + y$. Wlog due to symmetry, let us pick any one edge $e \in E(K_3)$ and its corresponding hyperplane $H \equiv x_1 - x_3 = 0 \in A_2$. Then:

$$G - e = P_2, \mathcal{A}_{G-e} = \mathcal{A} - H = \{x_1 - x_2 = 0, x_2 - x_3 = 0\}$$

$$G/e = C_2, \mathcal{A}_{G/e} = \mathcal{A}/H$$

Theorem 3.1 can be generalised to this setup via the observation that adding parallel edges / multiple hyperplanes also does not affect rank, hence the argument in the proof of Lemma 3.2 follows through. Let us verify this on the above

[5] Our definition still disallows loops.

$G - e$, G/e and hence illustrate how the recurrence also holds. Summing over subsets of size $2, 1, 0$:

$$\begin{aligned} T_{P_2}(x,y) &= (x-1)^0(y-1)^0 + 2(x-1)^1(y-1)^0 + (x-1)^2(y-1)^0 \\ &= 1 + 2(x-1) + (x-1)^2 \\ &= (x-1+1)^2 \\ &= x^2 \end{aligned}$$

$$\begin{aligned} T_{\mathcal{A}-H}(x,y) &= (x-1)^0(y-1)^0 + 2(x-1)^1(y-1)^0 + (x-1)^2(y-1)^0 \\ &= T_{P_2}(x,y) \end{aligned}$$

It is in the contraction / restriction that our new conditions arise.
Note that deleting one of a family of parallel edges does not disconnect any component. Hence :

$$\begin{aligned} T_{C_2}(x,y) &= (x-1)^0(y-1)^1 + 2(x-1)^0(y-1)^0 + (x-1)^1(y-1)^0 \\ &= (y-1) + 2 + (x-1) \\ &= x + y \end{aligned}$$

$\mathcal{A}/H = (\{x_1 = x_2 = x_3\}, \{(x_1 = x_2 = x_3, 2)\})$, in the ambient space H. We consider a $(\mathcal{B}, M')$ central if $\mathcal{B}$ is central. Thus :

$$\begin{aligned} T_{\mathcal{A}/H}(x,y) &= (x-1)^0(y-1)^1 + 2(x-1)^0(y-1)^0 + (x-1)^1(y-1)^0 \\ &= T_{C_2}(x,y) \end{aligned}$$

And so the recurrence holds.

However, we proved Theorem 5.3 for *all* arrangements, not just graphical arrangements. So let us verify it for an example that is not a braid subarrangement. Take $\mathcal{A}$ to be the Linial arrangement in $\mathbb{R}^3$. We checked that its Tutte polynomial is $x^2 + x + 1$. Wlog via symmetry, pick H to be $x_1 - x_3 = 1$.
So $\mathcal{A} - H = \{x_1 - x_2 = 1, x_2 - x_3 = 1\}$. All subarrangements are central. Hence, over subarrangements of size $2, 1, 0$:

$$\begin{aligned} T_{\mathcal{A}-H}(x,y)&(x-1)^0(y-1)^0 + 2(x-1)^1(y-1)^0 + (x-1)^2(y-1)^0 \\ &= 1 + 2(x-1) + (x-1)^2 \\ &= (x-1+1)^2 \\ &= x^2 \end{aligned}$$

Now let us look at $\mathcal{A}/H$. The ambient space is $H \equiv x_1 - x_3 = 1$. The restriction of $x_1 - x_2 = 1$ on H, by subtracting one equation from the other, is contained in $x_2 - x_3 = 0$. Similarly the restriction of $x_2 - x_3 = 1$ is contained in $x_1 - x_2 = 0$. However since $x_1 = x_2 = x_3$ cannot occur within H, the two restrictions do not intersect – i.e. $\mathcal{A}/H$ is two parallel lines in a plane, hence of rank 1. Thus, summing over each line and the empty subarrangement :

$$\begin{aligned} T_{\mathcal{A}/H}(x,y) &= 2(x-1)^0(y-1)^0 + (x-1)^1(y-1)^0 \\ &= 2 + (x-1) \\ &= x + 1 \end{aligned}$$

Hence the recurrence holds.

6 Relation to the characteristic polynomial

The Tutte polynomial of a hyperplane arrangement can be interpreted as an expectation on the characteristic polynomial of its subarrangements.

Theorem 6.1. *[1, Theorem 3.7] Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ and $0 \le t \le 1$ be a real number. Let $\mathcal{B}$ be a random subarrangement of $\mathcal{A}$ obtained by removing each hyperplane from $\mathcal{A}$ with probability t. Then*

$$\mathbb{E}[\chi(\mathcal{B}, q)] = q^{n-r}\overline{\chi}_{\mathcal{A}}(q, t)$$

The proof follows from Theorem 4.1 and its special case $t = 0$ (Athanasiadis's result exhibited in class). An example :
$\mathcal{A} = A_2$. $\mathcal{B}$ be a random subarrangement obtained as above. By symmetry, it is enough look at the possible cardinality of $\mathcal{B}$. Then :

- $P(|\mathcal{B}| = 3) = (1-t)^3$
- $P(|\mathcal{B}| = 2) = 3t(1-t)^2$
- $P(|\mathcal{B}| = 1) = 3t^2(1-t)$
- $P(|\mathcal{B}| = 0) = t^3$

Note that here, $n-r = 1$. Thus, from known finite field method for characteristic polynomial :

$$\begin{aligned}
LHS &= (1-t)^3 q(q-1)(q-2) + 3t(1-t)^2 q(q-1)^2 + 3t^2(1-t)q^2(q-1) + t^3q^3 \\
&= q[(1-t)^3(q-1)(q-2) + 3t(1-t)^2(q-1)^2 + 3t^2(1-t)q(q-1) + t^3q^2] \\
&= q(1-t)^2\left[(1-t)(q-1)(q-2) + 3t(q-1)^2 + \frac{3t^2q(q-1)}{1-t} + \frac{t^3q^2}{(1-t)^2}\right] \\
&= q(1-t)^2\left[(1-t)(q^2-3q+2) + 3t(q^2-2q+1) + \frac{3t^2q(q-1)}{1-t} + \frac{t^3q^2}{(1-t)^2}\right] \\
&= q(1-t)^2\left[q^2\Big(1-t+3t+\frac{3t^2}{1-t}+\frac{t^3}{(1-t)^2}\Big) + q\Big(3t-3-6t-\frac{3t^2}{1-t}\Big) + \Big(2-2t+3t\Big)\right] \\
&= q(1-t)^2\left[\frac{q^2}{(1-t)^2}\Big((1+2t)(1-t)^2 + 3t^2(1-t) + t^3\Big) - \frac{3q}{1-t}\Big((1+t)(1-t)+t^2\Big) + 2 + t\right] \\
&= q(1-t)^2\left[\frac{q^2}{(1-t)^2}\Big((1+2t)(1+t^2-2t) + 3t^2(1-t) + t^3\Big) - \frac{3q}{1-t} + 2 + t\right] \\
&= q(1-t)^2\left[\frac{q^2}{(1-t)^2}(1+2t+t^2+2t^3-2t-4t^2+3t^2-3t^3+t^3) - \frac{3q}{1-t} + 2 + t\right] \\
&= q(t-1)^2\left[\frac{q}{(t-1)^2} + \frac{3q}{t-1} + (t-1) + 3\right] \\
&= RHS \text{ ...from earlier computation of } \overline{\chi}_{A_2}(q,t)
\end{aligned}$$

7 Graded graphs and the $\mathcal{E}_n$ arrangement

In this section we shall do an interesting example, as exhibited in [1].

Definition 7.1. [6] *For a fixed finite set $A = \{a_1, a_2, .., a_k\} \subset \mathbb{Z}$ s.t. $a_1 < a_2 < .. < a_k$, $\mathcal{E}_n$ is a hyperplane arrangement in $\mathbb{R}^n$ containing the hyperplanes of the form:*

$$x_i - x_j = a_1, a_2, .., a_k \qquad\qquad 1 \leq i < j \leq n$$

Since it is a generalisation of the braid arrangement, we might expect that it has some natural connection with graphs that agrees with our known bijection. So we recall the definition of a graded graph from Section 3, and define :

Definition 7.2. *A graded graph is said to be* planted *if every connected component contains a vertex of height* 0.

Definition 7.3. *In a graded graph (V, E, h), the* slope $s(uv)$ *of an edge*[7] uv *is defined as $h(u) - h(v)$.*

Definition 7.4. *A be a finite set of integers. A graded graph is is called an A-graph if $\forall e \in E, s(e) \in A$.*

Given a planted graded graph $G = (V, E, h)$, $|V| = n$, define $\mathcal{A}_G$ as in Lemma 3.2. This is a central subarrangement of $\mathcal{E}_n$ on $A = \{s(e) : e \in E\}$, since $(h(1), h(2), .., h(n)) \in \cap\mathcal{A}_G$
Conversely given a central subarrangement $\mathcal{B} \subseteq \mathcal{E}_n$ on some A, for each $i < j$ and $a_t \in A$ s.t. $x_i - x_j = a_t \in \mathcal{B}$, we write down the equation $s(ij) = h(i) - h(j) = a_t$. Since $\mathcal{B}$ is central, this system of equations is consistent. We can thus obtain the heights of each vertex in terms of the smallest height in each connected component. Now forcing these smallest heights to be 0 "plants" the graph, giving us a planted graded A-graph.
By Lemma 3.2 and the above bijection of central subarrangements of $\mathcal{E}_n$ on A with planted graded A-graphs, since $r_{\mathcal{E}_n} = r_{A_{n-1}} = n - 1$, we get from the definition of coboundary polynomial:

Theorem 7.1. *[1, Proposition 4.10] For $\mathcal{E}_n$ on $A \subset \mathbb{Z}$:*

$$q\bar{\chi}_{\mathcal{E}_n}(x, y) = \sum_{\substack{G=(V,E,h) \\ \text{planted} \\ A-\text{graph}}} q^{c(G)}(t-1)^{|E|}$$

where $c(G)$ is the number of connected components in (V, E) and hence in G.

Let us conclude with an example of the bijection shown above. Note that $\mathcal{E}_n$ is not central so any central subarrangement is proper. Let $\mathcal{A}_G \subset \mathcal{E}_{10}$ be :

$$\begin{aligned} \{x_1 - x_2 &= 0, \\ x_1 - x_8 &= -2, \\ x_3 - x_7 &= 2, \\ x_3 - x_9 &= 3, \\ x_4 - x_7 &= -1, \\ x_5 - x_8 &= 2, \\ x_8 - x_{10} &= 0\} \end{aligned}$$

[6] This notation arises from ESA.
[7] recall : by our notation, this means $u < v$, so this is well-defined.

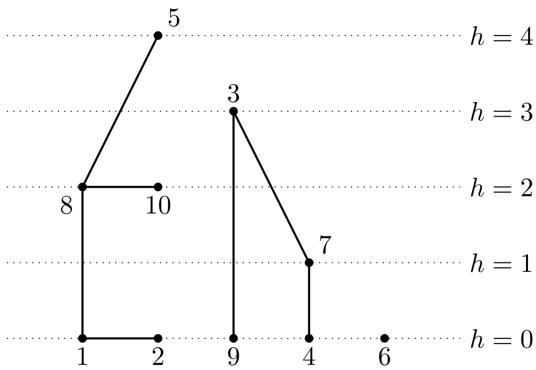
From this we know the edges on $V = [10]$. Knowing the connected components will help us "plant" the graph. We simply execute a search algorithm[8] : we find possible paths from 1. Vertices we reach are in the same component as 1, others are not. For the others, we restart the process from another vertex, say 3. Hence we get the components : $\{1, 2, 5, 8, 10\}, \{3, 4, 7, 9\}, \{6\}$. Now the equations in terms of heights are :

- $h(2) = h(1)$. $h(10) = h(8) = h(1) + 2, h(5) = h(8) + 2 \Rightarrow h(5) = h(1) + 4$
- $h(3) = h(7) + 2 = h(9) + 3 \Rightarrow h(7) = h(4) + 1 = h(9) + 1 \Rightarrow h(9) = h(4), h(3) = h(4) + 3$.

Thus the lowest vertices in each component are $\{1, 2\}, \{4, 9\}, \{6\}$. Forcing their heights to be 0, we get :

$$h^{-1}(0) = \{1, 2, 4, 6, 9\}$$
$$h^{-1}(1) = \{7\}$$
$$h^{-1}(2) = \{8, 10\}$$
$$h^{-1}(3) = \{3\}$$
$$h^{-1}(4) = \{5\}$$

This fixes G as in the diagram below. Clearly we shall obtain the exact same hyperplanes if we recover $\mathcal{A}_G$ from G by our rule.



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[8] depth-first or breadth-first, as illustrated in [4]

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