# Randomized Rounding for Uncapacitated Facility Location[1]

- In the *uncapacitated facility location* (UFL) problem, we are given a set of clients $C$, a set of facilities $F$, and a distance function $d$ on $C \cup F$ that satisfies the triangle inequality. We are also given *facility opening costs* $\{f_i\}_{i\in F}$. The task is to "open" a subset $S \subseteq F$ of facilities that minimizes the sum of the *facility cost* $\texttt{cost}_F(S) = \sum_{i\in S} f_i$ and the *connection cost* $\texttt{cost}_C(S) = \sum_{j\in C} d(j,S)$, where $d(j,S)$ is the smallest distance from $j$ to any point in $S$. This problem is NP-hard, and we look at an approximation algorithm for it that uses the primal and the dual linear programs for it, as follows. In the primal, variables $y_i$ denote whether $i \in S$, and $x_{ij}$ denotes whether client $j$ connects to facility $i$.

**Primal:**

$$\min \sum_{i\in F} f_i y_i + \sum_{j\in C}\sum_{i\in F} x_{ij} d(i,j)$$

$$\sum_{i\in F} x_{ij} \geq 1 \qquad \forall j \in C$$

$$y_i - x_{ij} \geq 0 \qquad \forall j \in C, i \in F$$

$$x_{ij} \geq 0, y_i \geq 0 \qquad \forall j \in C, i \in F$$

**Dual:**

$$\max \sum_{j\in C} v_j$$

$$\sum_{j\in C} w_{ij} \leq f_i \qquad \forall i \in F$$

$$v_j - w_{ij} \leq d(i,j) \qquad \forall i \in F, j \in C$$

$$v_j \geq 0, w_{ij} \geq 0 \qquad \forall i \in F, j \in C$$

We study a randomized algorithm that rounds a solution to the primal with the help of a solution to the dual. We first obtain an expected approximation ratio of $(1+3/e) < 2.11$, and then improve it to $(1+2/e) < 1.74$.

- We solve the primal LP to get $(x,y)$. We use some convenient notation:

$$\mathsf{lp}_F := \sum_{i\in F} f_i y_i \ ; \quad \mathsf{lp}_C := \sum_{j\in C}\sum_{i\in F} x_{ij} d(i,j) \ ; \quad C_j := \sum_{i\in F} d(i,j) x_{ij} \text{ for a client } j \in C$$

so that $\mathsf{lp} = \mathsf{lp}_F + \mathsf{lp}_C$ and $\mathsf{lp}_C = \sum_{j\in C} C_j$. We also solve the dual to get $(v,w)$. By complementary slackness[2], for any $i \in F, j \in C$, $x_{ij} > 0 \implies v_j - w_{ij} = d(i,j) \implies d(i,j) \leq v_j$. That is, in $(x,y)$, each client $j$ only uses facilities within distance $v_j$ of itself. We call this property $v$-*close*ness.

**Remark:** *We can avoid solving the dual. We only need that* $(x,y)$ *is* $v$*-close, and* $\sum_{j\in C} v_j \leq \mathsf{lp}$. *For this, it actually suffices to set, for each* $j \in C, v_j := \max\{d(i,j) \mid x_{ij} > 0\}$.

- ***A deterministic start.*** We first describe a simple deterministic rounding from the $v$-close fractional solution $(x,y)$ to a $3v$-close integral solution, i.e. $S \subseteq F$ such that $d(j,S) \leq 3v_j, \forall j \in C$. This is a "filtering" step, where we find a subset $D \subseteq C$ of clients that are far from each other. To do this, we pick the client $j_0$ with minimum $v_{j_0}$ into $D$, and put any clients that share facilities with $j_0$ into the *child set* of $j_0$, called $\mathsf{Chld}_{j_0}$, which we then discard. We repeat this until we run out of clients. To complete the algorithm, we open the cheapest facility in each $N(j_0) : j_0 \in D$.

[1] *Project report by Ankita Sarkar based on [1]. Last modified : March 10, 2022*
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[2] Assuming the reader knows this terminology from the course's lecture notes.

**procedure** FILTERING FOR UFL($F \cup C, \{f_i\}_{i \in F}, (x, y)$):
  $U \leftarrow C$, $D \leftarrow \emptyset$ ▷ *$U$ is the set of clients for which we have not decided*
  $\forall j \in C, N(j) \leftarrow \{i \in F \mid x_{ij} > 0\}$
  **while** $U \neq \emptyset$ **do**
    Pick $j_0 \in U$ with minimum $v_{j_0}$
    $D \leftarrow D + j_0$
    $\mathsf{Chld}_{j_0} \leftarrow \{j \in C \mid N(j) \cap N(j_0) \neq \emptyset\}$
    $U \leftarrow U \setminus \mathsf{Chld}_{j_0}$
  **return** $D$
For $D$ constructed above, $S \leftarrow \left\{i_0 = \operatorname{argmin}_{i \in N(j_0)} f_i \mid j_0 \in D\right\}$
**return** $S$

*Analysis.* Consider a client $j \in \mathsf{Chld}_{j_0}, j_0 \in D$. Let $i_0$ be the cheapest facility in $N(j_0)$. Then

$$d(j, S) \leq d(j, i_0) \leq d(j, j_0) + d(j_0, i_0) \leq d(j, i) + d(i, j_0) + d(j_0, i_0)$$

for some $i \in N(j) \cap N(j_0)$. By $v$-closeness of $(x, y)$, $d(j, i) \leq v_j$, and $d(i, j_0), d(j_0, i_0) \leq v_{j_0}$. Also by the filtering algorithm, since $j \in \mathsf{Chld}_{j_0}$, $v_{j_0} \leq v_j$. So $d(j, S) \leq 3v_j$. We can show that we already have a 4-approximation.

**Exercise:** *Show that* $\sum_{i \in S} f_i \leq \mathsf{lp}_F$*, and hence conclude that* $\mathtt{cost}_F(S) + \mathtt{cost}_C(S) \leq 4\mathsf{lp}$.

When we randomize, we use this 4-approximation as our worst-case "backup". That is, in most cases, we get $\mathbf{Exp}[d(j, S)] \leq C_j$ but, in a few bad cases, we rely on the deterministic bound $d(j, S) \leq 3v_j$.

- ***Randomization.*** Our randomized algorithm starts with the filtering process and obtains $D$. But instead of opening the cheapest facility in each $N(j_0) : j_0 \in D$, we open *one* $i_0$ in each $N(j_0)$ with probability proportional to $x_{ij_0}$. We also consider facilities outside all $N(j_0)$'s, i.e. $R := F \setminus \cup_{j_0 \in D} N(j_0)$. We independently open each $i \in R$ with probability $y_i$.

**procedure** RANDOMIZED ROUNDING FOR UFL($D \subseteq C$ from filtering, and $(x, y)$):
  $S \leftarrow \emptyset$
  **for** $j_0 \in D$ **do**
    Pick ***one*** $i_0 \in N(j_0)$ as per distribution $(x_{ij_0})_{i \in N(j_0)}$ ▷ $\sum_{i \in F} x_{ij_0} = 1$
    $S \leftarrow S + i_0$
  **for** $i \in R$ **do**
    With probability $y_i$, $S \leftarrow S + i$
  **return** $S$

*Analysis of Randomized Rounding.* For facility costs,

$$\mathbf{Exp}[\mathtt{cost}_F(S)] \leq \sum_{j_0 \in D} \mathbf{Exp}[f_{i_0}] + \sum_{i \in R} f_i \, \mathbf{Pr}[i \in S] = \sum_{j_0 \in D} \sum_{i \in N(j_0)} f_i x_{ij_0} + \sum_{i \in R} f_i y_i \leq \mathsf{lp}_F \quad (1)$$

To analyze the connection costs, fix a client $j$ such that $j \in \mathsf{Chld}_{j_0}, j_0 \in D$. Our algorithm always opens some $i_0 \in N(j_0)$, so by the same argument as the deterministic algorithm, $d(j,S)$ is at most $3v_j$. However, we hope that some facility in $N(j)$ is also open, allowing us to bound $d(j,S)$ by $v_j$ instead.

Using the pairwise disjointness of the sets $\{N(\ell)\}_{\ell\in D} \cup \{R\}$, we partition $N(j)$ into sets $\{S_\ell := N(j) \cap N_\ell\}_{\ell\in D} \cup \{R_j := N(j) \cap R\}$. For $\ell \in D$, $E_\ell$ be the event that we open some facility in $S_\ell$, and $p_\ell := \mathbf{Pr}[E_\ell] = \sum_{i\in S_\ell} x_{i\ell}$. Also, for each $i \in R_j$, $E_i$ be the event that $i$ is open, and $p_i := \mathbf{Pr}[E_i] = y_i$. Observe that all the events $\{E_\ell\}_{\ell\in D} \cup \{E_i\}_{i\in R_j}$ are pairwise independent. Let $B_j$ be the bad event that *none* of these events occur. Dropping the subscript for a fixed $j$,

$$\mathbf{Pr}[B] = \left(\prod_{\ell\in D}(1-p_\ell)\right)\left(\prod_{i\in R_j}(1-p_i)\right) \le e^{-\left(\sum_{\ell\in D} p_\ell + \sum_{i\in R_j} p_i\right)} \tag{2}$$

For ease of analysis, we make the following "completeness" assumption: $\forall j \in C, i \in N(j), x_{ij} = y_i$, i.e. clients use fractional facilities either as much as possible or not at all. This assumption can be made true by creating an equivalent instance where it holds.

**Exercise:** *Given an LP solution $(x,y)$ on the instance $\mathcal{I}$, give a polynomial time algorithm to construct a new instance $\mathcal{I}'$ and an LP solution $(x',y')$ on $\mathcal{I}'$ so that $\mathtt{cost}_{\mathcal{I}}(x,y) = \mathtt{cost}_{\mathcal{I}'}(x',y')$, and the latter is complete; and also if $(x,y)$ is $v$-close then so is $(x',y')$.*

Under this assumption, for $\ell \in D, i \in S_\ell, x_{i\ell} = y_i = x_{ij}$; and for $i \in R_j, y_i = x_{ij}$. So we get

$$\sum_{\ell\in D} p_\ell + \sum_{i\in R_j} p_i = \sum_{\ell\in D}\sum_{i\in S_\ell} x_{i\ell} + \sum_{i\in R_j} y_i = \sum_{i\in N(j)} x_{ij} = 1$$

So (2) gives $\mathbf{Pr}[B] \le 1/e$. Appealing to the backup cost when $B$ occurs, we get

$$\mathbf{Exp}[d(j,S)] \le \sum_{\ell\in D} \mathbf{Exp}[d(j,S_\ell) \mid E_\ell]p_\ell + \sum_{i\in R_j} \mathbf{Exp}[d(j,i) \mid E_i]p_i + 3v_j/e \tag{3}$$

For $\ell \in D$, conditioned on $E_\ell$ occurring, $i \in S_\ell$ is open with probability $x_{i\ell}/p_\ell$. So the first term above becomes

$$\sum_{\ell\in D} \mathbf{Exp}[d(j,S_\ell) \mid E_\ell]p_\ell = \sum_{\ell\in D}\left(\sum_{i\in S_\ell} d(j,i)x_{i\ell}/p_\ell\right)p_\ell = \sum_{\ell\in D}\sum_{i\in S_\ell} x_{ij}d(j,i)$$

where the last step is by our completeness assumption. Also, for $i \in R_j$, $\mathbf{Exp}[d(j,i) \mid E_i] \le d(j,i)$ and $p_i = y_i = x_{ij}$. So the RHS in (3) becomes at most

$$\sum_{\ell\in D}\sum_{i\in S_\ell} d(j,i)x_{ij} + \sum_{i\in R_j} d(j,i)x_{ij} + 3v_j/e \le \sum_{i\in N(j)} d(j,i)x_{ij} + 3v_j/e = C_j + 3v_j/e$$

by definition of $N(j)$. So summing over all $j \in C$,

$$\mathbf{Exp}[\mathtt{cost}_C(S)] \le \sum_{j\in C} C_j + (3/e)\sum_{j\in C} v_j = \mathsf{lp}_C + 3\mathsf{lp}/e \tag{4}$$

Adding facility costs from (1), $\mathbf{Exp}[\mathtt{cost}_F(S) + \mathtt{cost}_C(S)] \le \mathsf{lp}_F + \mathsf{lp}_C + 3\mathsf{lp}/e = (1+3/e)\mathsf{lp}$.

- In fact, one can do a little better. We can write, for a $j \in C$, $\mathbf{Exp}[d(j,S)] \leq \mathbf{Exp}[d(j,S) \mid \overline{B_j}](1-1/e) + \mathbf{Exp}[d(j,S) \mid B_j]/e$. So, intuitively, the $\mathsf{lp}_C$ term in (4) should be safe to multiply by $(1-1/e)$. To formalize this, we would need to show

  **Claim 1.** $\forall j \in C, \mathbf{Exp}[d(j,S) \mid \overline{B_j}] \leq C_j$

  This is **not** trivial to prove but, since it is intuitively believable, we believe it for the purpose of this note. Let us rewrite (4) accordingly.

$$\mathbf{Exp}[\mathtt{cost}_S(C)] \leq (1-1/e)\mathsf{lp}_C + \sum_{j \in C} \mathbf{Exp}[d(j,S) \mid B_j]/e \leq (1+3/e)\mathsf{lp} - \mathsf{lp}_C/e \qquad (4')$$

  We now pursue better bounds for $\mathbf{Exp}[d(j,S) \mid B_j]$ and use them with (4').

  **Exercise:** *Prove Claim 1, or read and understand its proof from [1].*

***Improved Filtering.*** Notice that our bounds on $\mathbf{Exp}[d(j,S) \mid \overline{B_j}]$ and $\mathbf{Exp}[\mathtt{cost}_F(S)]$ are ensured by the randomized rounding, as long as $\{N(\ell)\}_{\ell \in D}$ is pairwise disjoint. Contrarily, the bounds on $\mathbf{Exp}[d(j,S) \mid B_j]$ depend on the construction of $D$ itself. So we alter our filtering procedure to get a new $D$ that still has the above disjointness, but also gives better bounds on $\mathbf{Exp}[d(j,S) \mid B_j]$.

Observe that a $j_0 \in D$ never relies on the backup of $3v_{j_0}$; rather, because the randomized rounding opens some $i_0 \in N(j_0)$ with probability $x_{ij_0}$, $\mathbf{Exp}[d(j_0,S)] = C_{j_0}$, which could be much smaller than $v_{j_0}$. Connection costs of the form $C_{j_0} : j_0 \in D$ appear many times in our bounds; so perhaps we should pick vertices into $D$ by minimum $C_j$'s, rather than by minimum $v_j$'s? But, we do not want to abandon the benefits of $v$-closeness. So we strike a natural compromise: we pick vertices into $D$ by minimum $v_j + C_j$. That is, we replace Line 5 of the filtering algorithm with the following:

Pick $j_0 \in U$ with minimum $v_{j_0} + C_{j_0}$

*Analysis with Improved Filtering.* Our proof relies on the key lemma

**Lemma 1.** For any $j \in C$, $\mathbf{Exp}[d(j,S) \mid B_j] \leq 2v_j + C_j$.

By the lemma, $\sum_{j \in C} \mathbf{Exp}[d(j,S) \mid B_j] \leq \sum_{j \in C}(2v_j + C_j) = 2\mathsf{lp} + \mathsf{lp}_C$. So from (4') and (1), $\mathbf{Exp}[\mathtt{cost}_F(S) + \mathtt{cost}_C(S)] \leq \mathsf{lp}_F + (1-1/e)\mathsf{lp}_C + (2\mathsf{lp} + \mathsf{lp}_C)/e = (1+2/e)\mathsf{lp}$.

*Proof of Lemma 1.* Fix $j \in \mathsf{Chld}_{j_0}, j_0 \in D$. Drop the subscript from $B_j$. By the improved filtering, $v_{j_0} + C_{j_0} \leq v_j + C_j$. Also let $i_0$ be the facility in $N(j_0)$ that is opened during randomized rounding. Since $j \in \mathsf{Chld}_{j_0}$,$N(j) \cap N(j_0) \neq \emptyset$, so for some $i \in N(j) \cap N(j_0)$, $d(j,S) \leq d(j,i_0) \leq d(j,i) + d(i,j_0) + d(j_0,i_0)$. Since our target quantity is $2v_j + C_j$, we want to use $v$-closeness on two of these terms, and bound the remaining term using some connection cost. We proceed via two cases.

**Case 1.** This is the case where $\exists i \in N(j) \cap N(j)$, $d(i,j_0) \leq C_{j_0}$. Choosing this $i$ and using $v$-closeness, $d(j,i) + d(i,j_0) + d(j_0,i_0) \leq v_j + (C_{j_0} + v_{j_0}) \leq v_j + (C_j + v_j) = 2v_j + C_j$.

**Case 2.** This is the case where $\forall i \in N(j) \cap N(j_0), d(i,j) > C_{j_0}$. In this case, we show that

$$\mathbf{Exp}[d(j_0, i_0) \mid B] \le C_{j_0} \tag{5}$$

which gives, by linearity of expectation,

$$\mathbf{Exp}[d(j,i) + d(i,j_0) + d(j_0,i_0)] \le v_j + (v_{j_0} + C_{j_0}) \le 2v_j + C_j$$

We now prove (5). We know that $\mathbf{Exp}[d(j_0, i_0)] = C_{j_0}$. We are in the case where all facilities in $N(j) \cap N(j_0)$ are farther away from $j$ than this unconditioned expected cost. So conditioned on $B$ i.e. when they are all closed, the expected cost should not get worse. Formalizing this:

$$\begin{aligned}
C_{j_0} = \mathbf{Exp}[d(j_0, i_0)] &= \sum_{i \in N(j_0)} d(i, j_0) x_{ij_0} \\
&>_{(\textbf{Case 2})} \quad C_{j_0} \underbrace{\sum_{i \in N(j_0) \cap N(j)} x_{ij_0}}_{=:p} \; + \; (1-p) \sum_{i \in N(j_0) \setminus N(j)} d(i, j_0) x_{ij_0} / (1-p) \\
&= C_{j_0} p \; + \; (1-p) \sum_{i \in N(j_0) \setminus N(j)} d(i, j_0) \, \mathbf{Pr}[i_0 = i \mid N(j) \cap S = \emptyset] \\
&= C_{j_0} p + (1-p) \, \mathbf{Exp}[d(j_0, i_0) \mid B]
\end{aligned}$$

Rearranging, we get

$$(1-p) C_{j_0} > (1-p) \, \mathbf{Exp}[d(j_0, i_0) \mid B] \implies \mathbf{Exp}[d(j_0, i_0) \mid B] \le C_{j_0}$$

□

**Ponder This:** *What happens if we filter by picking clients with minimum* $C_j$*?*

## Notes

This algorithm appears in the paper [1] by Chudak and Shmoys. The initial deterministic idea, that yields a $4$-approximation, earlier appeared in a paper [2] by Shmoys, Tardos, and Aardal, and this is detailed in lecture note 5 of the course. [2] also used a different randomization to improve the $4$ to 3.16, and gave a simple derandomization for it.

## References


[1] F. A. Chudak and D. B. Shmoys. Improved approximation algorithms for the uncapacitated facility location problem. *SIAM Journal on Computing*, 33(1):1–25, 2003.

[2] D. B. Shmoys, É. Tardos, and K. Aardal. Approximation algorithms for facility location problems. In *Proceedings of the twenty-ninth annual ACM symposium on Theory of computing*, pages 265–274, 1997.